POINTWISE BEHAVIOR OF THE LINEARIZED BOLTZMANN EQUATION ON A TORUS

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ABSTRACT. We study the pointwise behavior of the linearized Boltzmann equation on torus for non-smooth initial perturbations. The result reveals both the fluid and kinetic aspects of this model. The fluid-like waves are constructed as part of the long-wave expansion in the spectrum of the Fourier modes for the space variable, and the time decay rate of the fluid-like waves depends on the size of the domain. We design a Picard-type iteration for constructing the increasingly regular kinetic-like waves, which are carried by the transport equations and have exponential time decay rate. The Mixture Lemma plays an important role in constructing the kinetic-like waves, and we supply a new proof of this lemma without using the explicit solution of the damped transport equations (compare with Liu-Yu’s proof [9, 12]).

1. Introduction

The Boltzmann equation for the hard sphere model reads

\[
\begin{align*}
\partial_t F + \xi \cdot \nabla_x F &= \frac{1}{\varepsilon} Q(F, F), \\
F(0, x, \xi) &= F_0(x, \xi),
\end{align*}
\]

where \(Q(\cdot, \cdot)\) is the so-called collision operator given by

\[
Q(g, h) = \frac{1}{2} \int_U \left[ -g(\xi)h(\xi) - g(\xi_+)h(\xi) + g(\xi')h(\xi'_+) + g(\xi'_+)h(\xi') \right] |(\xi - \xi_+) \cdot \Omega| d\xi_+ d\Omega
\]

with

\[
U = \{(\xi_+, \Omega) \in \mathbb{R}^3 \times S^2 : (\xi - \xi_+) \cdot \Omega \geq 0\}
\]

and

\[
\xi' = \xi - [(\xi - \xi_+) \cdot \Omega] \Omega, \quad \xi'_+ = \xi + [(\xi - \xi_+) \cdot \Omega] \Omega.
\]

Date: October 25, 2014.

2000 Mathematics Subject Classification. 35Q20; 82C40.

Key words and phrases. Boltzmann equation; Fluid-like wave; kinetic-like wave; Maxwellian states; Pointwise estimate.

The author would like to thank Professor Dr. Clément Mouhot for suggesting the subject, his encouragement and fruitful discussions. It is also a pleasure to thank I-Kun Chen, Seung-yeal Ha, Jin-Cheng Jiang, Chanwoo Kim, Hung-Wen Kuo, Chi-Kun Lin, Tai-Ping Liu and Se Eun Noh for stimulating discussion concerning this paper. This work is supported by the Tsz-Tza Foundation (Taiwan), National Science Council under the grant 102-2115-M-017-004-MY2 (Taiwan) and ERC grant MATKIT (European Union). Part of this work was written during the stay at Institute of Mathematics, Academia Sinica and Department of Mathematics, Stanford University; the author thanks Tai-Ping Liu for his kind hospitality.
Here $\varepsilon$ is the Knudsen number, the microscopic velocity $\xi \in \mathbb{R}^3$ and the space variable $x \in \mathbb{T}_{1/\varepsilon}^3$, the 3-dimensional torus with unit size of each side. In order to remove the parameter $\varepsilon$ from the equation, we introduce the new scaled variables:

$$
\tilde{x} = \frac{1}{\varepsilon} x, \quad \tilde{t} = \frac{1}{\varepsilon} t,
$$

then after dropping the tilde, the equation (1) becomes

$$
\begin{align*}
\partial_t F + \xi \cdot \nabla_x F &= Q(F, F), \\
F(0, x, \xi) &= F_0(x, \xi),
\end{align*}
$$

(2)

where $\mathbb{T}_{1/\varepsilon}^3$ denotes the 3-dimensional torus with size $1/\varepsilon$ of each side. The conservation laws of mass, momentum and energy can be formulated as

$$
\frac{d}{dt} \int_{\mathbb{T}_{1/\varepsilon}^3} \int_{\mathbb{R}^3} \left\{ 1, \xi, |\xi|^2 \right\} F(t, x, \xi) d\xi dx = 0.
$$

(3)

It is well-known that the Maxwellians are steady state solutions to the Boltzmann equation. Thus, it is natural to linearize the Boltzmann equation (2) around a global Maxwellian

$$
w(\xi) = \frac{1}{(2\pi)^{3/2}} \exp \left( -\frac{|\xi|^2}{2} \right),
$$

with the standard perturbation $f(t, x, \xi)$ to $w$ as

$$
F = w + w^{1/2} f.
$$

Then after substituting into (2) and dropping the nonlinear term, we have the linearized Boltzmann equation

$$
\begin{align*}
\partial_t f + \xi \cdot \nabla_x f &= 2w^{-1/2}Q(w, w^{1/2} f) = Lf, \\
F(0, x, \xi) &= I(x, \xi).
\end{align*}
$$

(4)

Here and below we define $f(t, x, \xi) = G_t^I I(x, \xi)$, i.e. $G_t^I$ is the solution operator (Green function) of the linearized Boltzmann equation (4). Assuming the initial density distribution function $F_0(x, \xi)$ has the same mass, momentum and total energy as the Maxwellian $w$, we can further rewrite the conservation laws (3) as

$$
\int_{\mathbb{T}_{1/\varepsilon}^3} \int_{\mathbb{R}^3} w^{1/2}(\xi) \left\{ 1, \xi, |\xi|^2 \right\} I(x, \xi) d\xi dx = 0.
$$

(5)

This means that the initial condition $I(x, \xi)$ satisfies the zero moments condition.

Before the presentation of the properties of the collision operator $L$, let us define some notations in this paper. For the microscopic variable $\xi$, we shall use $L^2_\xi$ to represent the classical Hilbert space with norm

$$
\|f\|_{L^2_\xi} = \left( \int_{\mathbb{R}^3} |f|^2 d\xi \right)^{1/2}.
$$
The Sobolev space of functions with all its $s$-th partial derivatives in $L^2_\xi$ will be denoted by $H^s_\xi$. The $L^2_\xi$ inner product in $\mathbb{R}^3$ will be denoted by $\langle \cdot, \cdot \rangle_\xi$ and the weighted sup norm is denoted by

$$\|f\|_{L^\infty_{\xi,\beta}} = \sup_{\xi \in \mathbb{R}^3} |f(\xi)|(1 + |\xi|)^\beta.$$ 

For the space variable $x$, we have the similar notations. In fact, $L^2_x$ is the classical Hilbert space with norm

$$\|f\|_{L^2_x} = \left( \frac{1}{|T_{1/\epsilon}^3|} \int_{T_{1/\epsilon}^3} |f|^2 \, dx \right)^{1/2},$$

and the Sobolev space of functions with all its $s$-th partial derivatives in $L^2_x$ will be denoted by $H^s_x$. We define the sup norm by

$$\|f\|_{L^\infty_x} = \sup_{x \in T_{1/\epsilon}^3} |f(x)|.$$ 

For simplicity of notations, hereafter, we abbreviate “$\leq C$” to “$\lesssim C$”, where $C$ is a positive constant depending only on fixed number.

**Proposition 1.** ([2, 5] Grad’s decomposition) The collision operator $L$ consists of a multiplicative operator $\nu(\xi)$ and an integral operator $K$: $Lf = -\nu(\xi)f + Kf$, where

$$Kf = \int_{\mathbb{R}^3} W(\xi, \xi_*) f(\xi_*) d\xi_*$$

is the linear integral operator with kernel

$$W(\xi, \xi_*) = \frac{2}{\sqrt{2\pi} |\xi - \xi_*|} \exp \left\{ - \frac{(|\xi|^2 - |\xi_*|^2)^2}{8 |\xi - \xi_*|^2} - \frac{|\xi - \xi_*|^2}{2} \right\} \exp \left\{ - \frac{|\xi|^2 + |\xi_*|^2}{4} \right\},$$

and the multiplicative operator $\nu(\xi)$ is given by

$$\nu(\xi) = \frac{1}{\sqrt{2\pi}} \left[ 2e^{-\frac{|\xi|^2}{4}} + 2(|\xi| + |\xi|^{-1}) \int_0^{|\xi|} e^{-\frac{u^2}{2}} \, du \right].$$

Moreover, for multiplicative operator $\nu(\xi)$, there exists a positive lower bound $\nu_0$ such that

$$\nu(\xi) \geq \nu_0, \quad \text{for all} \quad \xi \in \mathbb{R}^3,$$

and the derivatives of $\nu(\xi)$ in $\xi$ are bounded, i.e. for all multi index $\alpha$,

$$|\partial_\xi^\alpha \nu(\xi)| \leq C_\alpha.$$ 

The integral operator $K$ has smoothing properties in $\xi$, i.e.,

$$\|Kh\|_{H_\xi^1} \lesssim \|h\|_{L^2_\xi}, \quad \|Kh\|_{L^\infty_{\xi,\beta}} \lesssim \|h\|_{L^\infty_{\xi,\beta}},$$

for any $\beta \geq 0$. 
The integral operator $K$ can be decomposed into a singular part and a regular part, i.e., $K = K_s + K_r$, $K_s \equiv K_{s,D}$ and $K_r \equiv K_{r,D}$:

$$
\begin{align*}
K_s f &= \int_{\mathbb{R}^3} \chi \left( \left| \frac{\xi - \xi_*}{D\nu_0} \right| \right) W(\xi, \xi_*) f(\xi_*) d\xi_* , \\
K_r f &= K f - K_s f ,
\end{align*}
$$

where $\chi \in C_c^\infty(\mathbb{R})$ is nonnegative and satisfies $\chi(r) = 1$ for $r \in [-1, 1]$, and supp($\chi$) $\subset [-2, 2]$.

**Proposition 2.** ([12]) The singular part $K_s$ shares the same smoothing properties as $K$ and has strength of the order of the cut-off parameter $D$:

$$
\|K_s h\|_{H^s} \lesssim D\|h\|_{L^2} , \quad \|K_s h\|_{L^\infty_{\xi,\theta}} \lesssim D\|h\|_{L^\infty_{\xi,\theta}} .
$$

The regular part $K_r$ has better smoothing property in $\xi$: for all $s > 0$,

$$
\|K_r h\|_{H^s_{\xi}} \lesssim \|h\|_{L^2} .
$$

In order to estimate the Green function of the linearized Boltzmann equation in the next section, we need to recall the spectrum $\text{Spec}(\varepsilon k)$, $k \in \mathbb{Z}^3$, of the operator $-i\pi\varepsilon \xi \cdot k + L$ in the classical Hilbert space $L^2_{\xi}$.

**Proposition 3.** ([3, 10]) There exist $\delta > 0$ and $\tau = \tau(\delta) > 0$ such that

(i) For any $|\varepsilon k| > \delta$,

$$
\text{Spec}(\varepsilon k) \subset \{ z \in \mathbb{C} : \text{Re}(z) < -\tau \} .
$$

(ii) For any $|\varepsilon k| < \delta$, the spectrum within the region $\{ z \in \mathbb{C} : \text{Re}(z) > -\tau \}$ consisting of exactly five eigenvalues $\{\sigma_j(\varepsilon k)\}_{j=0}^4$,

$$
\text{Spec}(\varepsilon k) \cap \{ z \in \mathbb{C} : \text{Re}(z) > -\tau \} = \{\sigma_j(\varepsilon k)\}_{j=0}^4 ,
$$

and the corresponding eigenvectors $\{e_j(\varepsilon k)\}_{j=0}^4$. They have the expansions

$$
\sigma_j(\varepsilon k) = \sum_{n=0}^3 a_{j,n}(i|\varepsilon k|)^n + O(|\varepsilon k|) ,
$$

$$
e_j(\varepsilon k) = \sum_{n=0}^3 c_{j,n}(k/|k|)(i|\varepsilon k|)^n + O(|\varepsilon k|) ,
$$

where $a_{j,n}$ are constants, $a_{j,2} > 0$ and $\langle e_j(-\varepsilon k), e_l(\varepsilon k) \rangle_{\xi} = \delta_{jl}$, $0 \leq j, l \leq 4$.

(iii) The semigroup $e^{(-i\pi\varepsilon k + L)t}$ can be decomposed as

$$
e^{(-i\pi\varepsilon k + L)t} f = \Pi_\delta f + \chi_{|\varepsilon k| < \delta} \sum_{j=0}^4 e^{\sigma_j(\varepsilon k)t} \langle e_j(-\varepsilon k), f \rangle_{\xi} e_j(\varepsilon k) ,
$$

where $\chi_{\{\cdot\}}$ is the indicator function. Moreover, there exist $a(\tau) > 0$, $\alpha > 0$ such that

$$
\|\Pi_\delta\|_{L^2_{\xi}} \lesssim e^{-a(\tau)t} \quad \text{and} \quad e^{\sigma_j(\varepsilon k)t} \leq e^{-\alpha|\varepsilon k|^2t} \quad \text{for all} \quad 0 \leq j \leq 4 .
$$
The fluid behavior is studied by constructing the Green function represented as the Fourier series in the space variable $x$:

$$G_{\varepsilon}^t = \sum_{k \in \mathbb{Z}} \frac{1}{|T^3_{1/\varepsilon}|} e^{i\pi \varepsilon k \cdot x + (i\pi \varepsilon k + L)t}.$$

The analysis of the Green function is equivalent to the analysis of the spectrum of the operator $-i\pi \varepsilon \xi \cdot k + L$. Note that the spectrum includes five curves which bifurcate from the origin. The origin is the multiple zero eigenvalues of $L$, the operator at $k = 0$. The kernel of $L$ are the fluid variables and the fluid-like waves are constructed from these curves near the origin.

The kinetic aspect of the solution is described by the damped transport equation:

$$\partial_t g + \xi \cdot \nabla_x g + \nu(\xi)g = 0.$$

The operator $K$ is a smooth operator in the $\xi$ variable, and we will use this regularization property to design a Picard-type iteration for constructing the increasingly regular kinetic-like waves.

Once the kinetic-like waves and the fluid-like waves were constructed, the rest of the solution is sufficiently smooth and it has exponential time decay rate.

In the following, we describe our main result.

**Theorem 4.** Assuming that $\beta > \frac{3}{2}$, $I \in L^\infty_{\xi,\beta}$ with compact support in $x$ and satisfies the zero moments condition (5), then the solution $f = G_{\varepsilon}^t I$ of (4) can be decomposed into

$$f = G_{\varepsilon}^t I = G_{\varepsilon,F}^t I + G_{\varepsilon,K}^t I + G_{\varepsilon,R}^t I.$$

It consists of the fluid-like waves $G_{\varepsilon,F}^t I$, which is smooth in the space variable $x$ and the time decay rate depends on the size of the domain, i.e. there exist $\bar{a}, \delta, \delta_0 > 0$ such that

(i) If $\varepsilon > \delta$,

$$\| G_{\varepsilon,F}^t I \|_{L^\infty_x L^2_\xi} = 0,$$

(ii) If $\delta_0 < \varepsilon < \delta$,

$$\| G_{\varepsilon,F}^t I \|_{L^\infty_x L^2_\xi} \lesssim e^{-\bar{a}\varepsilon^2 t} \| I \|_{L^2_x L^2_\xi};$$

(iii) If $0 < \varepsilon < \delta_0$,

$$\| G_{\varepsilon,F}^t I \|_{L^\infty_x L^2_\xi} \lesssim \frac{1}{(1 + t)^{3/2}} e^{-\bar{a}\varepsilon^2 t} \| I \|_{L^1_x L^2_\xi};$$

the kinetic-like waves $G_{\varepsilon,K}^t I$, which is non smooth in the space variable, has the following exponential time decay: there exists $\nu_0 > 0$ such that

$$\| G_{\varepsilon,K}^t I \|_{L^\infty_x L^\infty_{\xi,\beta}} \lesssim (1 + t)^{5/2} e^{-\nu_0 t/2} \| I \|_{L^\infty_x L^\infty_{\xi,\beta}};$$

and the smooth remainder part $G_{\varepsilon,R}^t I$: there exists $C > 0$ such that

$$\| G_{\varepsilon,R}^t I \|_{H^2_x L^2_\xi} \lesssim e^{-Ct} \| I \|_{L^2_x L^2_\xi}.$$
The spectrum analysis of the Boltzmann equation was introduced by Ellis-Pinsky [3]. Moreover, Mouhot [16] gave the explicit coercivity estimates for the linearized Boltzmann operator. The analysis based on spectrum has been carried out by many authors. In particular, the exponential time decay rates for the Boltzmann equation with hard potentials on torus was firstly provided by Ukai [17]. The time-asymptotic nonlinear stability was obtained in [11, 18]. Using Nishida’s approach, [10] obtained the time-asymptotic equivalent of Boltzmann solutions and Navier-Stokes solutions. The functional spaces of these results are bounded in $L^2$ since the Fourier transform is used.

The Mixture Lemma plays an important role in constructing the kinetic-like waves. It states that the mixture of the two operators $S$ and $K$ in $M^j_1$ (see Section 4 below) transports the regularity in the microscopic velocity $\xi$ to the regularity in the space-time $(x, t)$. This idea was originally introduced by Liu-Yu [12, 13, 14, 15] to construct the Green function of the Boltzmann equation. In Liu-Yu’s papers [9, 12, 15], the proof of the Mixture Lemma relies on the explicit solution of the damped transport equation. However, in this paper, we introduce the differential operator $t\nabla_x + \nabla_\xi$ to avoid constructing the explicit solution, this operator commutes with the free transport operator and can transport the microscopic velocity regularity to the space regularity, this idea is similar to the iterated averaging lemma introduced by Gualdani-Mischler-Mouhot [6]. Moreover, this method will help us to consider more complicated problems, such as the Fokker-Planck equation or the Landau equation. Actually, we applied coercivity estimates [7, 16] to prove the Mixture Lemma for Landau equation with soft potentials [19]. Actually, the Mixture Lemma is similar in spirit to the well-known averaging lemma, see [1, 4, 8]. These two lemmas have been introduced independently and used for different purposes.

The pointwise description of the one-dimensional linearized Boltzmann equation with hard sphere was firstly provided by Liu-Yu [12], the fluid-like waves can be constructed by both complex and spectrum analysis, it reveals the dissipative behavior of the type of the Navier-Stokes equation as usually seems by the Chapman-Enskog expansion. The kinetic-like waves can be constructed by a Picard-type iteration and the Mixture Lemma. In this paper, we apply the similar ideas on torus, we can also construct the kinetic-like waves and the fluid-like waves, which are both time decay exponentially. Moreover, the decay rate of the fluid-like waves depends on the size of the domain.

The rest of the paper is organized as follows. In Section 2, we construct the Green function and the fluid-like waves of the linearized Boltzmann equation on torus. We apply the long wave short wave decomposition and the spectrum analysis to obtain time decay rate. In Section 3, we improve the estimate of the fluid-like waves. In Section 4, we design a Picard-type iteration for constructing the increasingly regular kinetic-like waves. Finally, we supply a new proof of the Mixture Lemma in the appendix.
2. LONG WAVE SHORT WAVE DECOMPOSITION

Consider the linearized Boltzmann equation

\[
\begin{aligned}
\partial_t f + \xi \cdot \nabla_x f &= Lf, \quad (t, x, \xi) \in (\mathbb{R}^+, T_{1/\varepsilon}^3, \mathbb{R}^3), \\
f(0, x, \xi) &= I(x, \xi),
\end{aligned}
\]

(13)

where \( I \) satisfies the zero moments condition (5). Hereafter, we will use just one index to denote the 3-dimensional sums with respect to the vector \( k = (k_1, k_2, k_3) \in \mathbb{Z}^3 \), hence we set

\[
\sum_{k \in \mathbb{Z}} = \sum_{(k_1, k_2, k_3) \in \mathbb{Z}^3}.
\]

Consider the Fourier series of the initial condition \( I \) in \( x \)

\[
\begin{aligned}
I(x, \xi) &= \sum_{k \in \mathbb{Z}} (\hat{I})_k(\xi)e^{i\varepsilon k \cdot x}, \\
(\hat{I})_k(\xi) &= \frac{1}{|T_{1/\varepsilon}^3|} \int_{T_{1/\varepsilon}^3} I(\cdot, \xi)e^{-i\varepsilon k \cdot x} dx,
\end{aligned}
\]

(14)

and rewrite the solution \( f(t, x, \xi) \) of (13) in terms of the Fourier series as

\[
\begin{aligned}
f(t, x, \xi) &= \sum_{k \in \mathbb{Z}} (\hat{f})_k(t, \xi)e^{i\varepsilon k \cdot x}, \\
(\hat{f})_k(t, \xi) &= \frac{1}{|T_{1/\varepsilon}^3|} \int_{T_{1/\varepsilon}^3} f(t, \cdot, \xi)e^{-i\varepsilon k \cdot x} dx.
\end{aligned}
\]

(15)

Note that the Fourier modes of (14)–(15) satisfy the following equations

\[
\begin{aligned}
\partial_t \hat{f}_k + i\pi \varepsilon \xi \cdot k \hat{f}_k - L\hat{f}_k &= 0, \\
\hat{f}_k(0, \xi) &= (\hat{I})_k,
\end{aligned}
\]

and can be solved explicitly as

\[
\hat{f}_k(t, \xi) = e^{(-i\pi \varepsilon k \cdot L)t}(\hat{I})_k(\xi).
\]

Hence the solution of (13) is given by

\[
\begin{aligned}
f(t, x, \xi) &= \sum_{k \in \mathbb{Z}} e^{i\varepsilon k \cdot x + (-i\pi \varepsilon k \cdot L)t}(\hat{I})_k(\xi) \\
&= \sum_{k \in \mathbb{Z}} \frac{1}{|T_{1/\varepsilon}^3|} \int_{T_{1/\varepsilon}^3} e^{i\varepsilon k \cdot (x-y) + (-i\pi \varepsilon k \cdot L)t} I(y, \xi) dy,
\end{aligned}
\]

which shows that the Green function \( \mathcal{G}_\varepsilon^t(x, \xi) \) is expressed explicitly as

\[
\mathcal{G}_\varepsilon^t(x, \xi) = \sum_{k \in \mathbb{Z}} \frac{1}{|T_{1/\varepsilon}^3|} e^{i\varepsilon k \cdot x + (-i\pi \varepsilon k \cdot L)t}.
\]
Note that when $\varepsilon \to 0$, the Green function $G^t_\varepsilon(x, \xi)$ will reduce to
\[
G_0^t(x, \xi) = \int_{\mathbb{R}^3} e^{i\eta \cdot x + (-i\pi \varepsilon \eta + L)t} d\eta.
\]

We decompose the Green function $G^t_\varepsilon(x, \xi)$ given by (16) into the long wave part $G^t_{\varepsilon,L}$ and the short wave part $G^t_{\varepsilon,S}$ respectively as
\[
G^t_{\varepsilon,L}(x, \xi) = \sum_{|k|<\delta} \frac{1}{|T^3_{1/\varepsilon}|} e^{i\pi k \cdot x + (-i\pi \varepsilon \xi \cdot k + L)t},
\]
\[
G^t_{\varepsilon,S}(x, \xi) = \sum_{|k|>\delta} \frac{1}{|T^3_{1/\varepsilon}|} e^{i\pi k \cdot x + (-i\pi \varepsilon \xi \cdot k + L)t}.
\]

The following long wave short wave analysis relies on the spectrum analysis (Proposition 3).

**Lemma 5.** (Short wave $G^t_{\varepsilon,S}$) For any $s > 0$, $I \in H^s_\varepsilon L^2_\xi$, we have
\[
\|G^t_{\varepsilon,S}I\|_{L^2_\varepsilon L^s_\xi} \leq \varepsilon^{-a(\tau)t} \|I\|_{L^2_\varepsilon L^s_\xi},
\]
\[
\|G^t_{\varepsilon,S}I\|_{H^2_\varepsilon L^s_\xi} \leq \varepsilon^{-a(\tau)t} \|I\|_{H^2_\varepsilon L^s_\xi}.
\]

**Proof.** Note that
\[
G^t_{\varepsilon,S}I = \sum_{|k|>\delta} e^{-i\pi \varepsilon \xi \cdot k + L)t} e^{i\pi k \cdot x} \hat{I}_k(\xi).
\]

For $L^2_\varepsilon L^s_\xi$ estimate, using Parseval’s theorem, we have
\[
\|G^t_{\varepsilon,S}I\|^2_{L^2_\varepsilon} = \sum_{|k|>\delta} |e^{(-i\pi \varepsilon \xi \cdot k + L)t} \hat{I}_k(\xi)|^2,
\]
by spectrum property (12) and Parseval’s theorem, we obtain
\[
\|G^t_{\varepsilon,S}I\|^2_{L^2_\varepsilon L^s_\xi} = \sum_{|k|>\delta} \|e^{(-i\pi \varepsilon \xi \cdot k + L)t} \hat{I}_k(\xi)\|^2_{L^2_\xi} \leq \varepsilon^{-2a(\tau)t} \sum_{|k|>\delta} \|\hat{I}_k(\xi)\|^2_{L^2_\xi}
\]
\[
= \varepsilon^{-2a(\tau)t} \int_{\mathbb{R}^3} \sum_{|k|>\delta} |\hat{I}_k(\xi)|^2 d\xi \leq \varepsilon^{-2a(\tau)t} \|I\|^2_{L^2_\varepsilon L^s_\xi}.
\]

Similarly, for high order estimates, we have
\[
\|G^t_{\varepsilon,S}I\|^2_{H^s_\varepsilon L^2_\xi} \leq \sum_{|k|>\delta} (1 + |\pi \varepsilon k|^2)^s |e^{(-i\pi \varepsilon \xi \cdot k + L)t} \hat{I}_k(\xi)|^2,
\]
and hence
\[
\|G^t_{\varepsilon,S}I\|^2_{H^s_\varepsilon L^2_\xi} \leq \varepsilon^{-2a(\tau)t} \sum_{|k|>\delta} (1 + |\pi \varepsilon k|^2)^s \|\hat{I}_k\|^2_{L^2_\xi} \leq \varepsilon^{-2a(\tau)t} \|I\|^2_{H^s_\varepsilon L^2_\xi}.
\]
This completes the proof of the lemma. \(\square\)
BOLTZMANN EQUATION

In order to study the long wave part $G^t_{\varepsilon,L}$, we need to decompose the long wave part as the fluid part $G^t_{\varepsilon,F}$ and the non-fluid part $G^t_{\varepsilon,L;\perp}$, i.e. $G^t_{\varepsilon,L} = G^t_{\varepsilon,F} + G^t_{\varepsilon,L;\perp}$, where

$$G^t_{\varepsilon,F} I = \sum_{|\varepsilon k| < \delta} \sum_{j=0}^{4} e^{\sigma_j(\varepsilon k)t} e^{i\varepsilon k \cdot x} \langle e_j(-\varepsilon k), \hat{I}_k \rangle \xi e_j(\varepsilon k),$$

(19)

$$G^t_{\varepsilon,L;\perp} I = \sum_{|\varepsilon k| < \delta} e^{i\varepsilon k \cdot x} \Pi_\delta \hat{I}_k.$$

**Lemma 6.** (Long wave $G^t_{\varepsilon,L}$) For any $s > 0$, $I \in L^2_x L^2_t$ and satisfies the zero moments condition (5), we have

$$\|G^t_{\varepsilon,L;\perp} I\|_{H^s_x L^2_t} \lesssim e^{-\alpha(\tau)t} \|I\|_{L^2_x L^2_t},$$

(20)

$$\|G^t_{\varepsilon,F} I\|_{H^s_x L^2_t} \lesssim e^{-\pi \varepsilon^2 t} \|I\|_{L^2_x L^2_t}.$$

**Proof.** For the non-fluid part, using (12) in Proposition 3 and Parseval’s theorem, we have

$$\|G^t_{\varepsilon,L;\perp} I\|_{H^s_x L^2_t}^2 \leq \sum_{|\varepsilon k| < \delta; |k| \neq 0} (1 + |\pi \varepsilon k|^2)^s |\Pi_\delta \hat{I}_k(\xi)|^2$$

$$\leq (1 + |\pi \delta|^2)^s \sum_{|\varepsilon k| < \delta; |k| \neq 0} |\Pi_\delta \hat{I}_k(\xi)|^2,$$

and hence

$$\|G^t_{\varepsilon,L;\perp} I\|_{H^s_x L^2_t}^2 \lesssim e^{-2\alpha(\tau)t} \|I\|_{L^2_x L^2_t}^2.$$

For the fluid-like waves, using (12) in Proposition 3, Parseval’s theorem and the zero moments condition (5), we have

$$\|G^t_{\varepsilon,F} I\|_{H^s_x L^2_t}^2 \lesssim \sum_{j=0}^{4} \left\| \sum_{|\varepsilon k| < \delta; |k| \neq 0} e^{\sigma_j(\varepsilon k)t} e^{i\varepsilon k \cdot x} \langle e_j(-\varepsilon k), \hat{I}_k \rangle \xi e_j(\varepsilon k) \right\|_{H^s_x L^2_t}^2$$

$$\lesssim (1 + |\pi \delta|^2)^s \sum_{j=0}^{4} \sum_{|\varepsilon k| < \delta; |k| \neq 0} |e^{\sigma_j(\varepsilon k)t}|^2 |\langle e_j(-\varepsilon k), \hat{I}_k \rangle \xi|^2$$

$$\lesssim \sum_{j=0}^{4} \sum_{|\varepsilon k| < \delta; |k| \neq 0} |\hat{I}_k|^2 \lesssim e^{-2\pi \varepsilon^2 t} \|I\|_{L^2_x L^2_t}^2.$$

This completes the proof of the lemma.

**Remark** (i) When $\varepsilon > \delta$, the long wave part vanishes, i.e. $G^t_{\varepsilon,L} = 0$.
(ii) For high order estimates of the short wave part require regularity in $x$, one needs higher regularity of $I$ to ensure the decay of $G_{\varepsilon,S}^t$ in time.

(iii) In order to remove the regularity assumption in $x$, we need a Picard-type iteration for constructing the increasingly regular kinetic-like waves in Section 4.

Combining Lemma 5 and Lemma 6, we deduce the following theorem.

**Theorem 7.** For any $I \in H_x^2 L_\xi^2$ satisfying the zero moments condition (5), we have the following exponential time decay estimate about the linearized Boltzmann equation (13)

$$\|G_{\varepsilon}^t I\|_{L_x^2 L_\xi^2} \lesssim e^{-\lambda_S t}\|I\|_{H_x^2 L_\xi^2} + e^{-\lambda_L t}\|I\|_{L_x^2 L_\xi^2}.$$ 

(i) If $\varepsilon > \delta$, then $\lambda_S = a(\tau)$ and $\lambda_L = \infty$.

(ii) If $\varepsilon \leq \delta$, then $\lambda_S = a(\tau)$ and $\lambda_L = \overline{a}\varepsilon^2$.

3. Fluid-like waves

In this section, we improve the estimate of the fluid-like waves. Recall the fluid-like waves of the Boltzmann equation (19)

$$G_{\varepsilon,F}^t I = \sum_{|\varepsilon k| < \delta} \sum_{j=0}^4 \varepsilon \sigma_j(\varepsilon k) t e^{i\varepsilon \xi} \langle e_j(-\varepsilon k), \hat{I}_k \rangle \xi e_j(\varepsilon k).$$

Applying the zero moments condition (5) and Cauchy-Schwarz inequality, we have

$$\|G_{\varepsilon,F}^t I\|_{L_x^2 L_\xi^2} \leq \sum_{j=0}^4 \sum_{|\varepsilon k| < \delta, |k| \neq 0} |\varepsilon \sigma_j(\varepsilon k)| t \|\hat{I}_k\|_{L_x^2 L_\xi^2}$$

$$\leq \frac{1}{|T_{1/\varepsilon}|} \sum_{j=0}^4 \sum_{|\varepsilon k| < \delta, |k| \neq 0} |\varepsilon \sigma_j(\varepsilon k)| \int_{T_{1/\varepsilon}} \|I\|_{L_x^2 L_\xi^2} dx$$

$$\lesssim \left( \int_{T_{1/\varepsilon}} \|I\|_{L_x^2 L_\xi^2} dx \right) \frac{1}{|T_{1/\varepsilon}|} \sum_{|\varepsilon k| < \delta, |k| \neq 0} e^{-\pi |\varepsilon k|^2 t}.$$  

Note that

$$\frac{1}{|T_{1/\varepsilon}|} \sum_{|\varepsilon k| < \delta} e^{-\pi |\varepsilon k|^2 t} = \frac{1}{t^{3/2}} \frac{1}{T_{1/\varepsilon}^3} \sum_{|\varepsilon k| < \delta t^{1/2}} e^{-\pi |\varepsilon k|^2} \to \frac{1}{t^{3/2}} \int_{B_{\delta t^{1/2}(0)}} e^{-\pi |y|^2} dy$$

as $\varepsilon \to 0$, this means for any $\alpha_0 > 0$ there exists $\delta_0 > 0$ such that if $\varepsilon < \delta_0$, we have

$$\frac{1}{|T_{1/\varepsilon}|} \sum_{|\varepsilon k| < \delta t^{1/2}} e^{-\pi |\varepsilon k|^2} < \alpha_0 + \int_{\mathbb{R}^3} e^{-\pi |y|^2} dy \equiv C_0.$$

Combining (22) and (23) together yields

$$\frac{1}{|T_{1/\varepsilon}|} \sum_{|\varepsilon k| < \delta} e^{-\pi |\varepsilon k|^2 t} \leq \frac{C_0}{(1 + t)^{3/2}}.$$
By (21) and (24), we get
\[ \| G^t_{\epsilon, F} I \|_{L^2_\xi} \lesssim \left( \frac{1}{|T^{3/4}|} \sum_{|k| < \delta, |k| \neq 0} e^{-\pi |k\epsilon|^2 t} \right) \| I \|_{L^1_1 L^1_\xi} \]
\[ \lesssim \left( \frac{1}{|T^{3/4}|} \sum_{|k| < \delta, |k| \geq 0} e^{-\pi |k\epsilon|^2 t} \right) e^{-\pi \epsilon^2 t} \| I \|_{L^1_1 L^2_\xi} \]
\[ \lesssim \frac{1}{(1 + t)^{3/2}} e^{-\pi \epsilon^2 t} \| I \|_{L^1_1 L^2_\xi}. \]

Under the above analysis, we have the following theorem.

**Theorem 8.** Assuming \( \epsilon < \delta \), \( I \in L^2_\xi \) with compact support in \( x \) and satisfying the zero moments condition (5), there exists \( \delta_0 > 0 \) such that if \( 0 < \epsilon < \delta_0 \), then we have
\[ \| G^t_{\epsilon, F} I \|_{L^2_\xi} \lesssim \frac{1}{(1 + t)^{3/2}} e^{-\pi \epsilon^2 t} \| I \|_{L^1_1 L^2_\xi}. \]

**Remark:** If \( \epsilon \to 0 \), the pointwise estimate of the fluid-like waves becomes
\[ \| G^t_{0, F} I \|_{L^2_\xi} \lesssim \frac{1}{(1 + t)^{3/2}} \| I \|_{L^1_1 L^2_\xi}. \]

We can compare the whole space case constructed by Liu-Yu in [12, 13].

## 4. KINETIC-LIKE WAVES AND REMAINDER PART

In this section, we will apply the kinetic decomposition and the Mixture Lemma to construct the kinetic-like waves and the remainder part. We rewrite the linearized Boltzmann equation (13) as
\[
\begin{aligned}
\partial_t f + \xi \cdot \nabla_x f + \nu(\xi) f - K_s f &= K_r f, \\
\end{aligned}
\]
\[ f(0, x, \xi) = I(x, \xi). \tag{25} \]

Now, we design a Picard type iteration, which treats the regular part \( K_r f \) as a source term. The \(-1\) order approximation of the linearized Boltzmann equation (25) is the damped transport equation
\[
\begin{aligned}
\partial_t h^{(-1)} + \xi \cdot \nabla_x h^{(-1)} + \nu(\xi) h^{(-1)} - K_s h^{(-1)} &= 0, \\
h^{(-1)}(0, x, \xi) &= I(x, \xi). \tag{26} \\
\end{aligned}
\]

Thus the difference \( f - h^{(-1)} \) will satisfy
\[
\begin{aligned}
\partial_t (f - h^{(-1)}) + \xi \cdot \nabla_x (f - h^{(-1)}) + \nu(\xi) (f - h^{(-1)}) &= K(f - h^{(-1)}) + K_r h^{(-1)}, \\
(f - h^{(-1)})(0, x, \xi) &= 0. \tag{27} \\
\end{aligned}
\]
Therefore we can define the zero order approximation $h^{(0)}$ by

\begin{align}
&\begin{cases}
\partial_t h^{(0)} + \xi \cdot \nabla_x h^{(0)} + \nu(\xi)h^{(0)} = K_r h^{(-1)}, \\
h^{(0)}(0, x, \xi) = 0.
\end{cases}
\end{align}

In general, we can define the $j$th order approximation $h^{(j)}$, $j \geq 1$, as

\begin{align}
&\begin{cases}
\partial_t h^{(j)} + \xi \cdot \nabla_x h^{(j)} + \nu(\xi)h^{(j)} = K h^{(j-1)}, \\
h^{(j)}(0, x, \xi) = 0.
\end{cases}
\end{align}

Therefore the solution $f = \mathcal{G}_t^I$ of the linearized Boltzmann equation (13) can be represented as a series

\[ f = h^{(-1)} + h^{(0)} + h^{(1)} + \cdots. \]

Let $\mathcal{S}^t$ and $\mathcal{O}^t$ denote the solution operators of the following equations,

\[ \begin{cases}
\partial_t g + \xi \cdot \nabla_x g + \nu(\xi)g = 0, \\
g(0, x, \xi) = g_0(x, \xi),
\end{cases} \]

and

\[ \begin{cases}
\partial_t j + \xi \cdot \nabla_x j + \nu(\xi)j - K_s j = 0, \\
j(0, x, \xi) = j_0(x, \xi),
\end{cases} \]

i.e.,

\[ g(t, x, \xi) = \mathcal{S}^t g_0(x, \xi), \quad j(t, x, \xi) = \mathcal{O}^t j_0(x, \xi). \]

By standard energy estimate, maximum principle and properties of the integral operator $K$ in (8), we have the following results about the operators $\mathcal{S}^t$ and $\mathcal{O}^t$.

**Lemma 9.** [12] For any $\beta \geq 0$, we have

\begin{align}
&\begin{cases}
\| \mathcal{S}^t g_0 \|_{L_x^2 L_\xi^2} \lesssim e^{-\nu_0 t} \| g_0 \|_{L_x^2 L_\xi^2}, \\
\| \mathcal{S}^t g_0 \|_{L_x^\infty L_\xi^\infty} \lesssim e^{-\nu_0 t} \| g_0 \|_{L_x^\infty L_\xi^\infty},
\end{cases}
\end{align}

and

\begin{align}
&\begin{cases}
\| \mathcal{O}^t j_0 \|_{L_x^2 L_\xi^2} \lesssim e^{-\nu_0 t/2} \| j_0 \|_{L_x^2 L_\xi^2}, \\
\| \mathcal{O}^t j_0 \|_{L_x^\infty L_\xi^\infty} \lesssim e^{-\nu_0 t/2} \| j_0 \|_{L_x^\infty L_\xi^\infty}.
\end{cases}
\end{align}

The following lemma gives the $L_x^2 L_\xi^2$ and $L_x^\infty L_\xi^\infty$ estimates of $h^{(j)}$.

**Lemma 10.** For $\beta \geq 0$, $j \geq -1$, we have

\begin{align}
&\| h^{(j)} \|_{L_x^2 L_\xi^2} \lesssim t^{j+1} e^{-\nu_0 t/2} \| I \|_{L_x^2 L_\xi^2},
\end{align}

and

\begin{align}
&\| h^{(j)} \|_{L_x^\infty L_\xi^\infty} \lesssim t^{j+1} e^{-\nu_0 t/2} \| I \|_{L_x^\infty L_\xi^\infty}.
\end{align}
Proof. We will prove this lemma by induction. The case $j = -1$ follows immediately from (26) and Lemma 9. For $j = 0$, by the definition of $h^{(0)}$ in (27) and Duhamel principle,

$$h^{(0)}(t, x, \xi) = \int_0^t S^{t-s} K_r \mathcal{O}^s I(\cdot, s_1) ds_1,$$

using proposition 2 and lemma 9, we have

$$\|h^{(0)}\|_{L^2_x L^2_\xi} \lesssim te^{-\nu_0 t/2} \|I\|_{L^2_x L^2_\xi}.$$

Next, assuming that the lemma holds for $j$, then by (7), (28) and (29), we have

$$\|h^{(j+1)}\|_{L^2_x L^2_\xi} = \left\| \int_0^t S^{t-s} (Kh^{(j)})(\cdot, s) ds \right\|_{L^2_x L^2_\xi} \lesssim \int_0^t e^{-\nu_0 (t-s)} e^{-\nu_0 s/2} s^{j+1} \|I\|_{L^2_x L^2_\xi} ds \lesssim t^{j+2} e^{-\nu_0 t/2} \|I\|_{L^2_x L^2_\xi}.$$

The estimates of $L^\infty_x L^\infty_\xi, \beta$ are similar and hence we omit the detail. This completes the proof of the lemma. $\square$

Next, we can define the kinetic decomposition

$$G^t_\varepsilon I = \sum_{j=-1}^4 h^{(j)} + \mathcal{R},$$

then the tail part $\mathcal{R}$ satisfies

$$\begin{cases}
\partial_t \mathcal{R} + \xi \cdot \nabla_x \mathcal{R} = L \mathcal{R} + Kh^{(4)}, \\
\mathcal{R}(0, x, \xi) = 0.
\end{cases}$$

Also, the kinetic-like waves and the remainder part can be defined as follows:

$$G^t_{\varepsilon, K} I = \sum_{j=-1}^4 h^{(j)}, \quad G^t_{\varepsilon, R} I = \mathcal{R} - G^t_{\varepsilon, F} I.$$

By Lemma 10, we have the pointwise estimate of the kinetic-like waves $G^t_{\varepsilon, K} I$

$$\|G^t_{\varepsilon, K} I\|_{L^\infty_x L^\infty_{\xi, \beta}} \lesssim (1 + t)^5 e^{-\nu_0 t/2} \|I\|_{L^\infty_x L^\infty_{\xi, \beta}}.$$

Now, let us apply Liu-Yu’s idea [12] to control the remainder part. Combining the long wave short wave decomposition (17) and the kinetic decomposition (32), we can define a function $u$ satisfies

$$u = G^t_{\varepsilon, S} I - \sum_{j=-1}^4 h^{(j)} = \mathcal{R} - G^t_{\varepsilon, L} I,$$
then from (18) and (31), there exists $C = C(\alpha, \nu_0) > 0$ such that
\[
\|u\|_{L^2_x L^2_\xi} = \left\| G_{\xi, S}^i I - \sum_{j=-1}^{4} h^{(j)} \right\|_{L^2_x L^2_\xi} \lesssim e^{-Ct} \|I\|_{L^2_x L^2_\xi}.
\]
This shows that $u$ belongs to the non-fluid part. For high order estimates, we need to calculate the equation of $u$, it is easy to see that $u$ solves the equation
\[
\begin{aligned}
\partial_t u + \xi \cdot \nabla_x u &= Lu + Kh^{(4)}, \\
u(0, x, \xi) &= u_0(x, \xi),
\end{aligned}
\] (33)
where the initial condition $u_0(x, \xi)$ is the long wave part of $-I(x, \xi)$. The energy estimate of (33) gives
\[
\frac{d}{dt} \|\nabla_x^2 u\|_{L^2_x L^2_\xi} \leq -C\|\nabla_x^2 u\|_{L^2_x L^2_\xi} + \|K(\nabla_x^2 h^{(4)})(\cdot, t)\|_{L^2_x L^2_\xi}.
\] (34)
This implies $L^\infty_x L^2_\xi$ estimate of $u$ if one can prove
\[
\int_0^t \|K(\nabla_x^2 h^{(4)})(\cdot, t-s)\|_{L^2_x L^2_\xi} ds \lesssim \|I\|_{L^2_x L^2_\xi}.
\] (35)
In order to proceed, define the $j^{th}$ Mixture operator as follows:
\[
\mathcal{M}_j^t f_0 = \int_0^t \int_{s_1}^{s_j} \cdots \int_{s_{j-1}}^{s_{j-2}} S^{t-s_1} K S^{s_1-s_2} K S^{s_2-s_3} K \cdots S^{s_{j-1}-s_j} K S^{s_j} f_0 ds_1 \cdots ds_j.
\] This form indicates that there are two essential mixing mechanisms:
(i) The mixing mechanism in $x$ is due to particles traveling in different velocity $\xi$. This is represented by the operator $S^t$.
(ii) The mixing mechanism in $\xi$ is due to the integral operator $K$.
Employing this definition, we have
\[
h^{(4)}(t, x, \xi) = \int_0^t \mathcal{M}_j^{t-s_0} K \mathcal{O} s_0 I(\cdot, s_0) ds_0.
\] (36)
It is nature to introduce the crucial Mixture Lemma.

**Lemma 11.** (Mixture Lemma [9, 12]) For any $f_0 \in L^2_x H^j_\xi$, $j = 1, 2$, we have
\[
\|\nabla_x^j \mathcal{M}_j^t f_0\|_{L^2_x L^2_\xi} \lesssim t^j e^{-2\nu_0 t/3} \|f_0\|_{L^2_x H^j_\xi}.
\] (37)
The Mixture Lemma states that the mixture of the two operators $S^t$ and $K$ in $\mathcal{M}_j^t$ transports the regularity in the microscopic velocity $\xi$ to the regularity of the space-time $(x, t)$. The proof of this lemma will be given in the appendix. We are now in the position
to prove (35), which is a consequence of the Mixture Lemma. Indeed, from (36) and (37), we have
\[
\| \nabla_x^2 h^{(4)}(\cdot, s) \|_{L^2_x L^2_\xi} \lesssim \int_0^s (s-s_0)^2 e^{-2\nu_0 (s-s_0)/3} \| \nabla_x^2 I \|_{L^2_x L^2_\xi} ds_0
\]
(38)
\[
\lesssim e^{-\nu_0 s/2} \int_0^s (s-s_0)^2 ds_0 \| I \|_{L^2_x L^2_\xi}
\]
\[
\lesssim s^3 e^{-\nu_0 s/2} \| I \|_{L^2_x L^2_\xi}.
\]
Using (7) and (38), we easily get
\[
\int_0^t \| K(\nabla_x^2 h^{(4)})(\cdot, t-s) \|_{L^2_x L^2_\xi} ds \lesssim \int_0^t \| K \|_{L^2_x L^2_\xi} \| \nabla_x^2 h^{(4)} \|_{L^2_x L^2_\xi} ds
\]
\[
\lesssim \| I \|_{L^2_x L^2_\xi} \int_0^t (t-s)^3 e^{-\nu_0 (t-s)/2} ds
\]
\[
\lesssim \| I \|_{L^2_x L^2_\xi}.
\]
Finally, from (34) and (35), we obtain
\[
\| u \|_{H^2_x L^2_\xi} \lesssim e^{-Ct} \| I \|_{L^2_x L^2_\xi}.
\]
Going back to the remainder part \( G^t_{\varepsilon, R} I \), we have
\[
\| G^t_{\varepsilon, R} I \|_{H^2_x L^2_\xi} = \| R - G^t_{\varepsilon, F} I \|_{H^2_x L^2_\xi} \leq \| u \|_{H^2_x L^2_\xi} + \| G^t_{\varepsilon, L; \perp} I \|_{H^2_x L^2_\xi} \lesssim e^{-Ct} \| I \|_{L^2_x L^2_\xi}.
\]
This completes the proof of Theorem 4.

5. Appendix: Proof of the Mixture Lemma

The purpose of this appendix is to give a short and direct proof of the Mixture Lemma without using the exact solution of the transport equation. This lemma was introduced by Liu-Yu [12, 13, 14, 15] where the proof relies on the explicit solution of the damped transport equation. In order to avoid constructing the explicit solution, we need to introduce the differential operator:
\[
\mathcal{D}_t = t \nabla_x + \nabla_\xi.
\]
It is important that \( \mathcal{D}_t \) commutes with the free transport operator:
\[
[\mathcal{D}_t, \partial_t + \xi \cdot \nabla_x] = 0,
\]
where \([A, B] = AB - BA\) is the commutator. We have the following estimates about operators \( \mathcal{D}_t \) and \( S^t \).

**Lemma 12.** For any \( f_0 \in L^2_x L^2_\xi \), there exists \( \eta_0 \) small enough such that
\[
\begin{cases}
\| \mathcal{D}_t S^t f_0 - S^t \nabla_\xi f_0 \|_{L^2_x L^2_\xi} \lesssim e^{-(\nu_0 - \eta_0)t} \| f_0 \|_{L^2_x L^2_\xi}, \\
\| \mathcal{D}_t^2 S^t f_0 - 2 \mathcal{D}_t S^t \nabla_\xi f_0 + S^t \nabla_\xi^2 f_0 \|_{L^2_x L^2_\xi} \lesssim e^{-(\nu_0 - \eta_0)t} \| f_0 \|_{L^2_x L^2_\xi}.
\end{cases}
\]
(39)
Remark: Although the integral operator \( K \) has smoothing property, it only allows us to differentiate with respect to \( \xi \) once (see (7) in Proposition 1), if we estimate the second order mixture operator \( \nabla^2_{\xi} M^t f_0 \), the trouble term \( \nabla^2_{\xi} K(\xi, \xi_\ast) \) appears. However, lemma 12 provides some cancelation properties and can overcome this difficulty.

**Proof.** We can check the following commutators
\[
[D_t, \nu(\xi)] h = \nabla_{\xi} \nu(\xi) h ,
\]
(40)
\[
[D^2_t, \nu(\xi)] h = 2[D_t, \nu(\xi)] D_t h + [D_t, [D_t, \nu(\xi)]] h = 2\nabla_{\xi} \nu(\xi) D_t h + \nabla^2_{\xi} \nu(\xi) h .
\]
For simplicity of notations, let \( g^k_s = D^k_t S^t \nabla^s_{\xi} f_0 \). One can check that \( g^1_s \) solves the equation
\[
\begin{cases}
\partial_t g^1_s + \xi \cdot \nabla_x g^1_s = -\nu(\xi) g^1_s - [D_t, \nu(\xi)] g^0_s , \\
g^1_s(0, x, \xi) = \nabla^s_{\xi} f_0(x, \xi) ,
\end{cases}
\]
(41)
and \( g^2_s \) solves the equation
\[
\begin{cases}
\partial_t g^2_s + \xi \cdot \nabla_x g^2_s = -\nu(\xi) g^2_s - 2[D_t, \nu(\xi)] g^1_s - [D_t, [D_t, \nu(\xi)]] g^0_s , \\
g^2_s(0, x, \xi) = \nabla^{s+1}_{\xi} f_0(x, \xi) .
\end{cases}
\]
(42)
To proceed, we define
\[
u^{(0)} = g^0_s = S_x f_0, \quad u^{(1)} = g^1_s - g^0_s, \quad u^{(2)} = g^2_s - 2g^1_s + g^0_s .
\]
The energy estimate for \( u^{(0)} (= S_x f_0) \) gives
\[
\| u^{(0)} \|_{L^2_{\xi} L^2_t}^2 + \int_0^t \| \nu^{1/2}(\xi) u^{(0)} \|_{L^2_{\xi} L^2_t}^2 ds \leq \| f_0 \|_{L^2_{\xi} L^2_t}^2 .
\]
We are now in the position to prove this lemma. For (39), one can calculate the equation of \( u^{(1)} \) by (41),
\[
\begin{cases}
\partial_t u^{(1)} + \xi \cdot \nabla_x u^{(1)} = -\nu(\xi) u^{(1)} - [D_t, \nu(\xi)] u^{(0)} , \\
u^{(1)}(0, x, \xi) = 0 .
\end{cases}
\]
(43)
Applying (6), (40) and energy estimate, there exists \( \eta_0 > 0 \) small enough such that
\[
\frac{1}{2} \frac{d}{dt} \| u^{(1)} \|_{L^2_{\xi} L^2_t}^2 + \| \nu^{1/2}(\xi) u^{(1)} \|_{L^2_{\xi} L^2_t}^2 \leq \eta_0 \| u^{(1)} \|_{L^2_{\xi} L^2_t}^2 + \eta_0^{-1} C_u \| u^{(0)} \|_{L^2_{\xi} L^2_t}^2 ,
\]
then we have
\[
\| u^{(1)} \|_{L^2_{\xi} L^2_t}^2 \lesssim e^{-\eta_0 - \eta_0 t} \| f_0 \|_{L^2_{\xi} L^2_t}^2 .
\]
This proves (39)$_1$. Similarly, for (39)$_2$, one can check that $u^{(2)}$ is a solution of the differential equation

$$\begin{align*}
\begin{cases}
\partial_t u^{(2)} + \xi \cdot \nabla_x u^{(2)} &= -\nu(\xi)u^{(2)} - 2[D_{t}, \nu(\xi)]u^{(1)} - [D_{t}, [D_{t}, \nu(\xi)]]u^{(0)}, \\
u^{(2)}(0, x, \xi) &= 0.
\end{cases}
\end{align*}$$

(44)

We can prove (39)$_2$ by the same argument. This completes the proof of the lemma. □

**Proof of the Mixture Lemma.** For $j = 1$, we can write down $\nabla_x M^1_1 f_0$ as follows:

$$\nabla_x M^1_1 f_0(x, \xi) = \int_0^t \int_0^{s_1} \nabla_x S^{s_1-s_2} K S^{s_1-s_2} f_0 \, ds_2 \, ds_1 \frac{\partial_t}{\partial_{s_1}} S^{s_1-s_2} f_0 \, ds_2 \, ds_1.$$

Note that $[\nabla_x, S^t] = 0$ and $[\nabla_x, W] = 0$, we can change the order of $(\nabla_x, S^t)$ and $(\nabla_x, W)$. To obtain the time integrability, we need to rewrite $\nabla_x M^1_1 f_0(x, \xi)$ as

$$\nabla_x M^1_1 f_0(x, \xi) = \int_0^t \int_0^{s_1} \int_{\mathbb{R}^6} S^{s_1-s_2} W(\xi, \xi_1) \nabla_x S^{s_1-s_2} W(\xi, \xi_2) \, ds_2 \, ds_1 \frac{\partial_t}{\partial_{s_1}} S^{s_1-s_2} f_0 \, ds_2 \, ds_1 \frac{\partial_t}{\partial_{s_1}} S^{s_1-s_2} f_0 \, ds_2 \, ds_1.$$

Using the fact $t\nabla_x = D_{t} - \nabla_\xi$ and integration by parts, we have

$$\nabla_x M^1_1 f_0(x, \xi) = \int_0^t \int_0^{s_1} \int_{\mathbb{R}^6} S^{s_1-s_2} W(\xi, \xi_1) \left(D_{s_1-s_2} - \nabla_\xi_1\right) S^{s_1-s_2} W(\xi, \xi_2) \, ds_2 \, ds_1 \frac{\partial_t}{\partial_{s_1}} S^{s_1-s_2} f_0 \, ds_2 \, ds_1 \frac{\partial_t}{\partial_{s_1}} S^{s_1-s_2} f_0 \, ds_2 \, ds_1 \frac{\partial_t}{\partial_{s_1}} S^{s_1-s_2} f_0 \, ds_2 \, ds_1 \frac{\partial_t}{\partial_{s_1}} S^{s_1-s_2} f_0 \, ds_2 \, ds_1.$$

where

$$I_1(f_0) = S^{s_1-s_2} W(\xi, \xi_1) D_{s_1-s_2} S^{s_1-s_2} W(\xi, \xi_2) S^{s_2} f_0 + S^{s_1-s_2} W(\xi, \xi_1) \nabla_\xi_2 S^{s_1-s_2} W(\xi, \xi_2) S^{s_2} f_0 + S^{s_1-s_2} \nabla_\xi_1 S^{s_1-s_2} W(\xi, \xi_2) D_{s_2} S^{s_2} f_0 + S^{s_1-s_2} \nabla_\xi_1 S^{s_1-s_2} W(\xi, \xi_2) S^{s_2} f_0.$$

By (39)$_1$, we have

$$||\nabla_x M^1_1 f_0||_{L^2_x L^2_\xi} \leq e^{-2\eta_1 t} \left(||f_0||_{L^2_x L^2_\xi} + ||\nabla_\xi f_0||_{L^2_x L^2_\xi} \left(\int_0^t \int_0^{s_1} \frac{1}{s_1} ds_2 \, ds_1 \right)\right).$$

$$= te^{-2\eta_1 t} \left(||f_0||_{L^2_x L^2_\xi} + ||\nabla_\xi f_0||_{L^2_x L^2_\xi} \right).$$
This proves the case \( j = 1 \). For \( j = 2 \), we can write down \( \nabla_x^2 M^f_2 f_0 \) purposely as follows:

\[
\nabla_x^2 M^f_2 f_0(x, \xi) = \int_T \int_{\mathbb{R}^6 \times 2} \frac{s_1 s_3}{s_1 s_3} \nabla_x^2 \left[ S^{t-s_1} W(\xi, \xi_1) S^{s_1-s_2} W(\xi_1, \xi_2) S^{s_2-s_3} W(\xi_2, \xi_3) S^{s_3-s_4} W(\xi_3, \xi_4) S^4 f_0 \right] d\Xi dS,
\]

where

\[
dS = ds_1 ds_2 ds_3 ds_4, \quad d\Xi = d\xi_1 d\xi_2 d\xi_3 d\xi_4,
\]

and

\[
T = [0, t] \times [0, s_1] \times \cdots \times [0, s_3], \quad 0 \leq s_4 \leq s_3 \leq s_2 \leq s_1 \leq t.
\]

In order to get time integrability, we need to write down \( \frac{s_1 s_3}{s_1 s_3} \) as

\[
\frac{s_1 s_3}{s_1 s_3} = \left[ (s_1 - s_2) + (s_2 - s_3) + (s_3 - s_4) + s_4 \right] \left[ (s_3 - s_4) + s_4 \right].
\]

Then we have

\[
\nabla_x^2 M^f_2 f_0(x, \xi) = \int_T \int_{\mathbb{R}^6 \times 2} \frac{1}{s_1 s_3} \left( J^1_2(f_0) + J^2_2(f_0) \right) d\Xi dS,
\]

where \( J^1_2(f_0) \) collects all the terms that each \( W \) differential with respect to \( \xi \) at most once, and \( J^2_2(f_0) \) collects all the terms that one of \( W \) differential with respect to \( \xi \) twice. More precisely, it can be expressed explicitly as

\[
J^2_2(f_0) = S^{t-s_1} W(\xi, \xi_1) S^{s_1-s_2} W(\xi_1, \xi_2) \left[ D_{s_2-s_3} S^{s_2-s_3} \right] \left[ \nabla_{\xi_3} W(\xi_2, \xi_3) \right] S^{s_3-s_4} W(\xi_3, \xi_4) S^4 f_0
\]

\[
+ S^{t-s_1} W(\xi, \xi_1) S^{s_1-s_2} W(\xi_1, \xi_2) S^{s_2-s_3} W(\xi_2, \xi_3) S^{s_3-s_4} W(\xi_3, \xi_4) S^4 f_0
\]

\[
+ S^{t-s_1} W(\xi, \xi_1) S^{s_1-s_2} W(\xi_1, \xi_2) S^{s_2-s_3} W(\xi_2, \xi_3) S^{s_3-s_4} W(\xi_3, \xi_4) S^4 f_0
\]

\[
+ 2 S^{t-s_1} W(\xi, \xi_1) S^{s_1-s_2} W(\xi_1, \xi_2) S^{s_2-s_3} W(\xi_2, \xi_3) \left[ D_{s_3-s_4} S^{s_3-s_4} \right] \left[ \nabla_{\xi_4} W(\xi_3, \xi_4) \right] S^4 f_0
\]

\[
+ S^{t-s_1} W(\xi, \xi_1) S^{s_1-s_2} W(\xi_1, \xi_2) S^{s_2-s_3} W(\xi_2, \xi_3) \left[ D_{s_3-s_4} S^{s_3-s_4} \right] W(\xi_3, \xi_4) S^4 f_0.
\]

The estimate of \( J^1_2(f_0) \) is similar to the case \( j = 1 \),

\[
\left\| \int_{\mathbb{R}^6 \times 2} J^1_2(f_0) d\Xi \right\|_{L^2 L^2_\xi} \lesssim e^{-2\eta t \frac{\xi^2}{s}} \| f_0 \|_{L^2 H^2_\xi}.
\]

The first two terms of \( J^2_2(f_0) \) can be estimated by \( (39)_1 \) and the last three terms can be estimated by \( (39)_2 \), then

\[
\left\| \int_{\mathbb{R}^6 \times 2} J^2_2(f_0) d\Xi \right\|_{L^2 L^2_\xi} \lesssim e^{-2\eta t \frac{\xi^2}{s}} \| f_0 \|_{L^2 L^2_\xi}.
\]

Therefore we have derived the estimate

\[
\| \nabla_x^2 M^f_2 f_0 \|_{L^2 L^2_\xi} \lesssim e^{-2\eta t \frac{\xi^2}{s}} \| f_0 \|_{L^2 H^2_\xi} \left( \int_T \frac{1}{s_1 s_3} dS \right) \lesssim t^2 e^{-2\eta t \frac{\xi^2}{s}} \| f_0 \|_{L^2 H^2_\xi}.
\]

This completes the proof of the Mixture Lemma. \( \square \)


References


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