GLOBAL IN TIME ESTIMATES FOR THE SPATIALLY HOMOGENEOUS LANDAU EQUATION WITH SOFT POTENTIALS

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Abstract. This paper deals with some global in time a priori estimates of the spatially homogeneous Landau equation for soft potentials $\gamma \in [-2, 0)$. For the first result, we obtain the estimate of weak solutions in $L^\alpha_t L^{3-\varepsilon}_v$ for $\alpha = \frac{2(3-\varepsilon)}{3(2-\varepsilon)}$ and $0 < \varepsilon < 1$, which is an improvement over estimates by Fournier-Guerin [10]. For the second result, we have the estimate of weak solutions in $L^\infty_t L^p_v$, $p > 1$, which extends part of results by Fournier-Guerin [10] and Alexandre-Liao-Lin [1]. As an application, we deduce some global well-posedness results for $\gamma \in [-2, 0)$. Our estimates include the case $\gamma = -2$, which is the key point in this paper.

1. Introduction

1.1. The Landau equations. We consider the spatially homogeneous Landau equation in dimension three for soft potentials. This equation of kinetic physics, also called Fokker-Planck-Landau equation, has been derived from the Boltzmann equation when the grazing collisions prevail in the gas. It describes the evolution of the density function $f_t(v)$ of particles having the velocity $v \in \mathbb{R}^3$ at time $t \geq 0$:

$$\frac{\partial f_t(v)}{\partial t} = Q(f_t, f_t)(v),$$

with the collision operator

$$Q(f_t, f_t)(v) = \nabla_v \cdot \left\{ \int a(v - v_*) \left[ f_t(v_*) \nabla f_t(v) - f_t(v) \nabla v_* f_t(v_*) \right] dv_* \right\},$$

where $a(v)$ is a symmetric non-negative matrix, depending on a parameter $\gamma \in [-3, 1]$, $a(v) = |v|^{\gamma+2}P(v)$.

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and $P(v)$ is the 3 by 3 matrix

$$P(v) = I_3 - \frac{v \otimes v}{|v|^2}.$$ 

This leads to the usual classification in terms of hard potentials $\gamma > 0$, Maxwellian molecules $\gamma = 0$, soft potentials $\gamma \in (-2, 0)$, very soft potentials $\gamma \in (-3, -2)$ and Coulomb potential $\gamma = -3$. Just as for the Boltzmann equation, little is known for soft potentials, i.e. $\gamma < 0$, and even less for very soft potentials, i.e. $\gamma < -2$. In particular, $\gamma = -3$ corresponds to the important Coulombic interaction in plasma physics. Unfortunately, it is also the most difficult case to study. However, the Landau equation can be derived from the Boltzmann equation with $\gamma \in (-3, -1)$. Note the fact that the more $\gamma$ is negative, the more the Landau equation is physically interesting. In this paper, we focus on soft potentials $\gamma \in [-2, 0)$.

For a given non-negative initial data $f_{in}(v)$, we shall use the notations

$$m(f_{in}) = \int f_{in}(v)dv, \quad e(f_{in}) = \frac{1}{2} \int f_{in}(v)|v|^2dv, \quad H(f_{in}) = \int f_{in}(v) \log f_{in}(v)dv,$$

for the initial mass, energy and entropy. It is classical that if $f_{in}(v) \geq 0$ and $m(f_{in}), e(f_{in}), H(f_{in})$ are finite, then $f_{in}(v)$ belongs to

$$L \log L(\mathbb{R}^3) = \left\{ f(v) \in L^1(\mathbb{R}^3) : \int |f(v)|| \log(|f(v)|)|dv < \infty \right\}.$$

The solution of the Landau equation (1) satisfies, at least formally, the conservation of mass, momentum and kinetic energy, that is, for any $t \geq 0$,

$$\int f_{t}(v)\varphi(v)dv = \int f_{in}(v)\varphi(v)dv, \quad \text{for} \quad \varphi(v) = 1, v, |v|^2. \quad (2)$$

Especially, we define

$$m(f_{t}) = \int f_{t}(v)dv, \quad e(f_{t}) = \frac{1}{2} \int f_{t}(v)|v|^2dv, \quad H(f_{t}) = \int f_{t}(v) \log f_{t}(v)dv.$$

Another fundamental a priori estimate is the entropy estimate, that is, the solution satisfies, at least formally, for any $t > 0$,

$$\frac{d}{dt} H(f_{t}) + D(f_{t}) = 0, \quad (3)$$

where

$$D(f_{t}) = 2 \int \int dv dv_{*} a(v - v_{*})(\nabla - \nabla_{*})\sqrt{f_{t}(v)f_{t}(v_{*})}(\nabla - \nabla_{*})\sqrt{f_{t}(v)f_{t}(v_{*})}, \quad (4)$$

here we use the notation $Mxx = x^{T}Mx$, the standard quadratic form, where $M$ is a matrix and $x$ is a vector. Note that $a(v - v_{*})$ is a non-negative matrix, this implies $D(f_{t}) \geq 0$, and hence the entropy is decreasing

$$\int f_{t}(v) \log f_{t}(v)dv \leq \int f_{in}(v) \log f_{in}(v)dv.$$
From the entropy equation (3), it is natural to introduce the weighted Fisher information as follows:

\[ I_s(f_t) = \int \left| \langle v \rangle^s \nabla \sqrt{f_t} \right|^2 dv. \]  

where \( \langle v \rangle \equiv (1 + |v|^2)^{\frac{1}{2}} \). For further use, we define the moment of order \( s \), for \( s \geq 0 \),

\[ \|f_t\|_{L_s^1(\mathbb{R}^3)} = \int |f_t(v)|\langle v \rangle^s dv \equiv M_s(f_t). \]

And we set (the notation \( \Sigma \) is omitted)

\[ b_i(z) = \partial_j a_{ij}(z), \quad c(z) = \partial_{ij} a_{ij}(z), \]

\[ \bar{a}_{ij} = a_{ij} * f_t, \quad \bar{b}_i = b_i * f_t, \quad \bar{c} = c * f_t. \]

If \( \gamma > -3 \), we have

\[ a_{ij}(z) = P_{ij}(z)|z|^{\gamma + 2}, \quad b_i(z) = -2|z|^{\gamma + 2} \frac{z_i}{|z|^2}, \quad c(z) = -2(\gamma + 3)|z|^\gamma. \]

Finally, let us introduce the weak formulation of (1): for any test function \( \varphi : \mathbb{R}^3 \rightarrow \mathbb{R} \),

\[ \int \varphi(v)f_T(v)dv = \int \varphi(v)f_{in}(v)dv + \int_0^T dt \int \int f_t(v)f_t(v_*)L\varphi(v,v_*)dvdv_*, \]

where the operator \( L \) is defined by

\[ L\varphi(v,v_*) = \frac{1}{2} \sum_{i,j=1}^3 a_{ij}(v-v_*)\partial_{ij}^2 \varphi(v) + \sum_{i=1}^3 b_i(v-v_*)\partial_i \varphi(v). \]

1.2. Review of previous works. The theory of the spatially homogeneous Landau equation for hard potentials was studied in great details by Desvillettes-Villani [7, 8], while the particular case of Maxwellian molecules \( \gamma = 0 \) can be found in Villani [14].

Concerning soft potentials, for the spatially inhomogeneous case, Guo [9] constructed global in time classical solutions near Maxwellians, and Chen-Desvillettes-He [6] studied the smoothing effects for classical solutions. For the spatially homogeneous case, by using a probabilistic approach, Guerin [12] studied the existence of a measure solution for \( \gamma \in (-1, 0) \). Still by probabilistic approach, Fournier-Guerin [10] studied the uniqueness of such a weak solution for soft potentials, moreover, they have global existence for \( \gamma \in (-2, 0) \) and local existence for \( \gamma \in (-3, -2) \). For the Coulomb potential case \( \gamma = -3 \), Arseñe-Peskov [2] studied the local existence of weak solutions and Fournier [11] considered the local well-posedness result for such solutions. For the \( L^2 \) a priori estimate, Alexandre-Liao-Lin [1] constructed the global estimate for \( \gamma \in (-3, 0) \), but their results need weighted spaces and need smallness assumption of the initial condition for \( \gamma \in [-3, -2] \).

All these results listed above give a priori estimates of solutions in some \( L^p \) spaces, globally if \( \gamma \in (-2, 0) \), either locally or globally but in weighted space and need smallness assumption of the initial condition if \( \gamma \in [-3, -2] \). This means that \( \gamma = -2 \) is the critical case in their estimates.
In this paper, we construct some global in time a priori estimates for the spatially homogeneous Landau equation, our results include the case $\gamma = -2$. Moreover, those results do not need in weighted space and smallness assumption of the initial condition.

The Fisher information is an important quantity for the Landau equation (1), which has already been successfully used in the convergence towards equilibrium for hard potentials and Maxwellian molecules [8, 14]. In this paper, we construct the weighted Fisher information for $\gamma = -2$ (proposition 11), then apply it to $L^\alpha([0, T], L^{3-\varepsilon})$ estimate in theorem 1. Moreover, it can also improve the temporal growth of the $L^p$ estimate in theorem 2.

For simplicity of notations, hereafter, we abbreviate “$\leq C$ ” to “$\lesssim C$ ”, where $C$ is a positive constant depending only on fixed parameter.

1.3. Main result I. We establish $L^\alpha([0, T], L^{3-\varepsilon})$ a priori estimate of the spatially homogeneous Landau equation (1) for $\gamma \in [-2, 0)$, here $\alpha > 1$ and $0 < \varepsilon < 1$, which is an improvement over estimates by Fournier-Guerin [10] (p.2555, proposition 10). In Fournier-Guerin’s work, they obtained global (in time) a priori estimate of weak solutions in $L^1_t L^{3-\varepsilon}_v$ for $\gamma \in (-2, 0)$. However, our result has the following improvements:

(i) We have better time integrability $L^\alpha_t$ ($\alpha > 1$).

(ii) Our estimate includes the case $\gamma = -2$.

The temporal growth of this result is polynomial if $\gamma \in (-2, 0)$ and exponential if $\gamma = -2$.

**Theorem 1.** Let $\gamma \in [-2, 0)$ be fixed, $0 < \varepsilon < 1$ and $s > 0$. Consider a weak solution $f_t(v)$ of (1) with initial datum $f_{in}(v) \in L^1_t(\mathbb{R}^3) \cap L \log L(\mathbb{R}^3)$ for some $q = \frac{-3(\gamma-s)(2-\varepsilon)}{\varepsilon} > 0$. Then, at least formally, $f_t(v) \in L^\alpha([0, T], L^{3-\varepsilon})$ with $\alpha = \frac{2(3-\varepsilon)}{3(2-\varepsilon)}$. More precisely,

$$\int_0^T \| f_t \|_{L^{3-\varepsilon}}^\alpha dt \lesssim C_T,$$

where

$$C_T \lesssim \begin{cases} 
(1 + T)^{1 + \frac{2\varepsilon}{3(2-\varepsilon)(2+\gamma)}}, & \gamma \in (-2, 0), \\
\exp \{ CT^z \}, & z = 2 + \frac{8\varepsilon}{3(2+s)(2-\varepsilon) - 6\varepsilon}, & \gamma = -2,
\end{cases}$$

for some constant $C > 0$.

The proof of this theorem is based on the modified entropy production estimate (by choosing $\beta(x) = (1+x) \log(1+x)$ in proposition 10), it was firstly introduced by Desvillettes-Villani [7] for hard potentials $\gamma > 0$, later on, Fournier-Guerin [10] applied it to $L^1([0, T], L^{3-\varepsilon})$ estimate for soft potentials $\gamma \in (-2, 0)$. In this paper, we can control $\| f_t \|^{\alpha}_{L^{3-\varepsilon}}$ by using the Hardy-Littlewood-Sobolev inequality for $\gamma \in (-2, 0)$. However, $\gamma = -2$ is the critical case, the Fisher information estimate can overcome this difficulty.

1.4. Main result II. The second result of this paper is devoted to $L^\infty([0, T], L^p)$ a priori estimate of the spatially homogeneous Landau equation (1) for $\gamma \in [-2, 0)$, $1 < p < \infty$, which extends part of results by Fournier-Guerin [10] (p. 2559, proposition 11) and Alexandre-Liao-Lin [1].
In Alexandre-Liao-Lin’s work [1], they constructed a global (in time) $L^\infty([0,T],L^2)$ a priori estimate for $\gamma \in (-3,0)$, but when $\gamma \in (-3,-2]$, their estimate needs a weighted space and extra smallness assumption of the initial condition. On the other hand, in Fournier-Guerin’s work [10], they constructed local (in time) $L^\infty([0,T],L^p)$ solutions for $\gamma \in (-3,-2]$. However, in this paper, we have global (in time) $L^\infty([0,T],L^p)$ a priori estimate for $\gamma \in [-2,0)$, $1 < p < \infty$, the weighted space and smallness assumption of the initial condition are not necessary in my result.

The temporal growth of this result is exponential if $\gamma \in (-2,0)$. When $\gamma = -2$, we need high order moment assumption $L^3_\lambda(\mathbb{R}^3)$, $\lambda > 2$ of the initial condition, and the temporal growth is double exponential. Moreover, the temporal growth reduces to exponential if we have very high order ($\lambda > 6$) moment assumption.

**Theorem 2.** Let $\gamma \in [-2,0)$ be fixed, $\lambda > 2$ and $1 < p < \infty$. Consider a weak solution $f_t(v)$ of (1) with initial datum $f_{in}(v) \in L^1_\lambda(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ for $\gamma \in (-2,0)$ and $f_{in}(v) \in L^2_\lambda(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ for $\gamma = -2$. Then, at least formally, $f_t(v) \in L^\infty([0,T],L^p)$. More precisely,

$$\|f_t\|_{L^p}^p \lesssim C_t,$$

where $C_t \lesssim e^{Ct}$ for $\gamma \in (-2,0)$. For $\gamma = -2$, we have $C_t \lesssim \exp\{Ct^{\frac{2}{\lambda-2}}\}$ if $2 < \lambda < 5$, $C_t \lesssim \exp\{Ct^{2}\}$ if $5 \leq \lambda \leq 6$ and $C_t \lesssim \exp\{Ct^{\frac{2(\lambda-5)}{\lambda-2}}\}$ if $\lambda > 6$, $C > 0$ is a positive constant.

For $\gamma \in (-2,0)$, we use Alexandre-Liao-Lin’s idea [1] to construct general $L^p$ theorem. However, $\gamma = -2$ is the critical case, in order to overcome this difficulty, we need a function decomposition and apply uniform bound of entropy to get a parameter of freedom. Moreover, if we have extra moment assumption of the initial condition, the Fisher information estimate (Proposition 11) plus bootstrap procedure can improve the temporal growth.

1.5. **Application.** As an application, we deduce some well-posedness results. Let us recall the existence and uniqueness for the spatially homogeneous Landau equation (1) with $\gamma \in (-3,0)$. First, Fournier-Guerin [10] have shown the uniqueness holds in $L^\infty([0,T],L^1_2) \cap L^1([0,T],\mathcal{J}_\gamma)$, where $\mathcal{J}_\gamma$ is the space of probability measures $f_t$ on $\mathbb{R}^3$ such that

$$J_\gamma(f_t) \equiv \sup_{v \in \mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^\gamma f_t(v_*)dv_* < \infty.$$

Note that $J_\gamma(f_t)$ has the following estimate: there exists a constant $C > 0$ such that

$$J_\gamma(f_t) \leq \|f_t\|_{L^1} + C\|f_t\|_{L^p},$$

as soon as $p > \frac{3}{3+\gamma}$. This implies the uniqueness also holds for

$$L^\infty([0,T],L^1_2) \cap L^1([0,T],L^p).$$
For the existence result, suppose that the initial condition \( f_{in}(v) \in L^1_{2}(\mathbb{R}^3) \cap L \log L(\mathbb{R}^3) \), Villani [15] has shown the existence of weak solution \( f_t(v) \in L^{\infty}([0,T], L^2_2) \), with constant energy and nonincreasing entropy.

Combining our a priori estimates and existence/uniqueness results listed above, we have the following well-posedness results (compare with Fournier-Guerin’s results [10]):

Corollary 3 is a consequence of theorem 1.

**Corollary 3.** Let \( \gamma \in (-2, 0) \) be fixed, \( 0 < \varepsilon < \frac{3(2+\gamma)}{3+\gamma} \) and \( s > 0 \). Assuming that the initial condition \( f_{in}(v) \in L^1_{q}(\mathbb{R}^3) \cap L \log L(\mathbb{R}^3) \) for some \( q = \frac{3(\gamma-s)(2-s)}{\varepsilon} > 0 \). Then the Landau equation (1) has a unique weak solution in \( L^{\infty}([0,T], L^3_{1}) \cap L^{\alpha}([0,T], L^{3-\varepsilon}) \) with \( \alpha = \frac{2(3-s)}{3(2-s)} \).

More precisely,

\[
\int_{0}^{T} \| f_t \|^\alpha_{L^{3-\varepsilon}} dt \lesssim (1 + T)^{\frac{\varepsilon}{\gamma-s}} \exp(CT^2).
\]

**Remark** If \( \gamma = -2 \), theorem 1 only implies existence result. We can not obtain uniqueness result since the space integrability is not enough.

**Corollary 4.** Assume that \( \gamma = -2 \), \( 0 < \varepsilon < 1 \) and \( s > 0 \). Let the initial condition \( f_{in}(v) \in L^1_{q}(\mathbb{R}^3) \cap L \log L(\mathbb{R}^3) \) for some \( q = \frac{3(2+s)(2-s)}{\varepsilon} > 0 \). Then there exists a weak solution of the Landau equation (1) in \( L^{\infty}([0,T], L^3_{2}) \cap L^{\alpha}([0,T], L^{3-\varepsilon}) \) with \( \alpha = \frac{2(3-s)}{3(2-s)} \).

More precisely,

\[
\int_{0}^{T} \| f_t \|^\alpha_{L^{3-\varepsilon}} dt \lesssim \exp \{ CT^2 \},
\]

where \( C \) is a constant and the polynomial power \( z \) was defined in theorem 1.

Corollary 5 is a consequence of theorem 2. We have well-posedness result for \( \gamma = -2 \) in this corollary.

**Corollary 5.** Let \( \gamma \in [-2,0) \) be fixed, \( \lambda > 2 \) and \( p > \frac{3}{3+\gamma} \). Assuming that the initial condition \( f_{in}(v) \in L^1_{2}(\mathbb{R}^3) \cap L^{p}(\mathbb{R}^3) \) for \( \gamma \in (-2,0) \) and \( f_{in}(v) \in L^1_{1}(\mathbb{R}^3) \cap L^{p}(\mathbb{R}^3) \) for \( \gamma = -2 \). Then the Landau equation (1) has a unique weak solution in \( L^{\infty}([0,T], L^p) \). More precisely,

\[
\| f_t \|^p_{L^p} \lesssim C_t,
\]

where \( C_t \lesssim e^{Ct} \) for \( \gamma \in (-2,0) \). For \( \gamma = -2 \), we have \( C_t \lesssim \exp \exp \{ C t^{\frac{2}{\lambda-5}} \} \) if \( 2 < \lambda < 5 \), \( C_t \lesssim \exp \exp \{ C t^{\frac{2\lambda}{5}} \} \) if \( 5 \leq \lambda \leq 6 \) and \( C_t \lesssim \exp \exp \{ C t^{2} \} \) if \( \lambda > 6 \), \( C > 0 \) is a positive constant.

1.6. **Plan of the paper.** The paper is organized as follows: We first list some important properties of the spatially homogeneous Landau equation in section 2. Then we prove theorem 1 \( (L^1_t L^3_{2-\varepsilon} \text{ estimate}) \) in section 3. Next, we prove theorem 2 \( (L^\infty_t L^p_v \text{ estimate}) \) in section 4.
2. Preliminaries

In this section, we list some important properties of the spatially homogeneous Landau equation (1). First, we have basic moments and entropy estimates.

**Proposition 6.** Let $\gamma \in [-2,0)$ be fixed. Consider a weak solution $f_i(v)$ of (1) with non-negative initial condition

$$f_{\text{in}}(v) \in L^1_2(\mathbb{R}^3) \cap L\log L(\mathbb{R}^3),$$

then

$$M_s(f_i) \lesssim 1 \quad \text{for} \quad 0 \leq s \leq 2, \quad \text{and} \quad H(f_i) \lesssim 1.$$

For high order moments estimate, we recall the following result for $\gamma \in (-2,0)$ in [16] (Section 2.4, p.73).

**Proposition 7.** Let $\gamma \in (-2,0)$ be fixed and $T > 0$. Consider a weak solution $f_i(v)$ of (1) with $t \in [0,T]$. Assuming that the initial condition $f_{\text{in}}(v) \in L^1_s(\mathbb{R}^3)$, $s > 2$, then

$$M_s(f_i) \lesssim (1 + t) \quad \text{for} \quad t \in [0,T].$$

On the other hand, if $\gamma = -2$, we have

**Proposition 8.** Assume that $\gamma = -2$, $T > 0$. Let us consider a weak solution $f_i(v)$ of (1) with $t \in [0,T]$. Assuming that the initial condition $f_{\text{in}}(v) \in L^1_s(\mathbb{R}^3)$, $s > 2$, then for $t \in [0,T]$

$$M_s(f_i) \lesssim \begin{cases} (1 + t) & \text{if} \quad 2 < s < 5, \\ (1 + t)^{\frac{s-2}{4}} & \text{if} \quad s \geq 5. \end{cases}$$

**Proof.** We still use the abstract notation $\gamma$. Let us recall from [7] the basic equation for the moments $M_s(f_i)$:

$$\frac{d}{dt} M_s(f_i) = s \int \int f_i(v) f_i(v_*) |v - v_*|^\gamma \langle v \rangle^{s-2}$$

$$\times \left\{ -2|v|^2 + 2|v_*|^2 + (s-2)\left(\frac{|v|^2|v_*|^2 - (v \cdot v_*)^2}{1 + |v|^2}\right) \right\} dv dv_*$$

$$= s \int \int f_i(v) f_i(v_*) |v - v_*|^\gamma \langle v \rangle^{s-2}$$

$$\times \left\{ -2\langle v \rangle^2 + 2\langle v_* \rangle^2 + (s-2)\left(\frac{|v|^2|v_*|^2 - (v \cdot v_*)^2}{1 + |v|^2}\right) \right\} dv dv_* .$$

(7)

By the H"older inequality,

$$\int \int f_i(v) f_i(v_*) |v - v_*|^\gamma \langle v \rangle^{s-2} \langle v_* \rangle^2 dv dv_*$$

$$\leq \left( \int \int f_i(v) f_i(v_*) |v - v_*|^\gamma \langle v \rangle^s dv dv_* \right)^{\frac{(s-2)}{s}} \left( \int \int f_i(v) f_i(v_*) |v - v_*|^\gamma \langle v_* \rangle^s dv dv_* \right)^{\frac{2}{s}}$$
\[ = \int \int f_t(v) f_t(v_\ast) |v - v_\ast|^\gamma \langle v \rangle^s dv dv_\ast, \]

then (7) becomes
\[ \frac{d}{dt} M_s(f_t) = s(s-2) \int \int f_t(v) f_t(v_\ast) |v - v_\ast|^\gamma \langle v \rangle^{s-2} \left( \frac{|v|^2 |v_\ast|^2 - (v \cdot v_\ast)^2}{1 + |v|^2} \right) dv dv_\ast, \]

and using the fact
\[ |v|^2 |v_\ast|^2 - (v \cdot v_\ast)^2 \leq |v||v_\ast||v - v_\ast|^2, \]

we find
\[ \frac{d}{dt} M_s(f_t) \leq s(s-2) \int \int f_t(v) f_t(v_\ast) |v - v_\ast|^\gamma+2 \langle v \rangle^{s-4} |v||v_\ast| dv dv_\ast. \]

It is easy to see that if \( \gamma = -2, \)
\[ \frac{d}{dt} M_s(f_t) \leq s(s-2) M_1(f_t) M_{s-3}(f_t) \lesssim M_{s-3}(f_t). \]

We shall present the estimates of \( M_s(f_t) \) for \( 2 < s < 5 \) and \( s \geq 5 \) separately. If \( 2 < s < 5, \) it is easy to see that
\[ \frac{d}{dt} M_s(f_t) \lesssim M_{s-3}(f_t) \lesssim 1, \]

and hence \( M_s(f_t) \lesssim (1 + t). \) On the other hand, if \( s > 5, \) by the Hölder inequality and the finiteness of \( M_2(f_t), \) we have
\[ M_{s-3}(f_t) = \int f_t(v) \langle v \rangle^{s-3} dv \leq M_s^{\frac{s-3}{s-2}}(f_t) M_2^{\frac{3}{s-2}}(f_t) \lesssim M_s^{\frac{s-3}{s-2}}(f_t), \]

then
\[ \frac{d}{dt} M_s(f_t) \lesssim M_s^{\frac{s-3}{s-2}}(f_t), \]

and hence \( M_s(f_t) \lesssim (1 + t)^{\frac{s-2}{s-3}}. \) This completes the proof of the proposition. \( \square \)

Let us introduce a result on the ellipticity of the matrix \( \bar{a}^{ft}. \)

**Proposition 9.** ([1, 7]) Let \( \gamma \in [-2, 0) \) be fixed. Consider a weak solution \( f_t(v) \) of (1) with non-negative initial condition
\[ f_{in}(v) \in L_2^1(\mathbb{R}^3) \cap L \log L(\mathbb{R}^3), \]

then there exists a constant \( C_{coer} > 0 \) such that
\[ \bar{a}_{ij}^{ft} \xi_i \xi_j \geq C_{coer} \langle v \rangle^{\gamma} |\xi|^2, \text{ for all } \xi \in \mathbb{R}^3. \]

We have the chain rule for the Landau equation as follows:

**Proposition 10.** ([7]) Let \( f_t(v) \) be a weak solution of the Landau equation and \( \beta \) be a \( C^1 \) function with \( \beta(0) = 0, \) then at least formally,
\[ \frac{d}{dt} \int \beta(f_t(v)) dv = - \int \bar{a}^{ft} \nabla f_t(v) \nabla f_t(v) \beta''(f_t(v)) dv - \int \bar{c}^{ft} \phi_\beta(f_t(v)) dv, \]
where
\[ \phi''_\beta(x) = x\phi'''_\beta(x), \quad \text{and} \quad \phi_\beta(0) = 0. \]

This property may help us to estimate the Landau equation by choosing suitable \( \beta \).

Finally, let us introduce the entropy production estimate (weighted Fisher information) of the spatially homogeneous Landau equation (1) for \( \gamma = -2 \). This proposition is the key point in this paper.

**Proposition 11.** (Weighted Fisher information) If \( \gamma = -2 \). Let us consider the Landau equation (1) with initial datum \( f_{\text{in}} \in L^1_d(\mathbb{R}^3) \cap L \log L(\mathbb{R}^3), \ d > 6 \). Then, at least formally,
\[ \int_0^T I_{-1}(f_t)dt \lesssim C_T, \]
where \( C_T \lesssim \exp\{CT^\kappa\}, C > 0 \) is a positive constant and \( \kappa = \frac{2(d-2)}{(d-6)} \).

**Proof.** Recall the entropy equation (3)
\[ \frac{d}{dt}H(f_t) + \mathcal{D}(f_t) = 0, \]
where
\[ \mathcal{D}(f_t) = \int \pi^t(v)|\nabla \sqrt{f_t(v)}|^2dv + \int v^t(v)f_t(v)dv \equiv F_1(f_t) - F_2(f_t). \]
Using the ellipticity of \( \pi^t \) in proposition 9, we easily get
\[ F_1(f_t) \geq C_{\text{coer}} \int \langle v \rangle^{-2} |\nabla \sqrt{f_t(v)}|^2dv \gtrsim I_{-1}(f_t). \]
For \( F_2(f_t) \), we have
\[ F_2(f_t) \lesssim \int \int |v - v_*|^{-2} f_t(v)f_t(v_*)dvdv_* \]
\[ = \int \int_{|v - v_*| > R} + \int \int_{|v - v_*| \leq R} |v - v_*|^{-2} f_t(v)f_t(v_*)dvdv_* \]
\[ \equiv F_{21}(f_t) + F_{22}(f_t). \]
It is easy to see that
\[ F_{21}(f_t) \leq R^{-2} M_0^2(f_t) \lesssim R^{-2}. \]
Applying the Cauchy-Schwarz inequality, we have obtained
\[ F_{22}(f_t) \leq \int \int_{|v - v_*| \leq R} (|v - v_*|^{-1} f_t(v))(|v - v_*|^{-1} f_t(v_*)dvdv_* \]
\[ \leq \int \int_{|v - v_*| \leq R} |v - v_*|^{-2} f_t^2(v)dvdv_* \]
\[ \lesssim R \int f_t^2(v)dv. \]
Consider the following decomposition:
\[
\int f_t^2(v) dv = \int f_t^2(v) 1_{\{\log f_t(v) < A\}} dv + \int f_t^2(v) 1_{\{\log f_t(v) \geq A\}} dv
\equiv F_{221}(f_t) + F_{222}(f_t),
\]
where \(A = A(t)\) is a parameter to be chosen later. It is obvious that
\[
F_{221}(f_t) \leq e^A M_0(f_t) \lesssim e^A.
\]
For \(F_{222}(f_t)\), note that \(\log f_t(v) \geq A\), we have the following decomposition
\[
f_t^2(v) \leq (\sqrt{f_t(v)} \langle v \rangle^{-1})^{\frac{6}{2d}} (f_t(v) \log f_t(v))^{\frac{1}{2d}} \langle f_t(v) \rangle^{\frac{d}{2d}} \frac{A}{\sqrt{2}}.
\]
where \(\frac{3}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 2, \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1\) and \(-\frac{6}{q_1} + \frac{d}{q_3} = 0\), one can choose \(\frac{1}{q_1} = \frac{1}{2}, \frac{1}{q_2} = \frac{1}{a}, \frac{1}{q_2} = \frac{1}{2} - \frac{1}{a}\) and \(d = 3a, d > 6\), by the Hölder inequality and the Sobolev embedding theorem,
\[
F_{222}(f_t) \leq A^{-\frac{d-6}{2d}} H^{\frac{d-6}{2d}}(f_t) M_d^3(f_t) \\| \langle v \rangle^{-1} \sqrt{f_t(v)} \\|_L^3
\lesssim A^{-\frac{d-6}{2d}} M_d^3(f_t) (\| \nabla(\langle v \rangle^{-1} \sqrt{f_t(v)}) \|_L^2 + \| \langle v \rangle^{-1} \sqrt{f_t(v)} \|_L^2)^3
\lesssim A^{-\frac{d-6}{2d}} M_d^3(f_t) (I_{-1}^2(f_t) + 1).
\]
Combining (12)–(17), we have
\[
F_2(f_t) \lesssim R^{-2} + Re^A + RA^{-\frac{d-6}{2d}} M_d^3(f_t) (I_{-1}^2(f_t) + 1).
\]
The estimate of \(F_2(f_t)\) depends on the size of \(I_{-1}(f_t)\). First, if \(I_{-1}(f_t) < 1\), then (choose \(R = A = 1\))
\[
F_2(f_t) \lesssim M_d^3(f_t).
\]
On the other hand, if \(I_{-1}(f_t) \geq 1\), we can optimize \(RA^{-\frac{d-6}{2d}} M_d^3(f_t) I_{-1}^2(f_t)\) and \(R^{-2}\) with respect to \(R\), i.e.
\[
R = A^{\frac{d-6}{2d}} M_d^{-\frac{1}{2}}(f_t) I_{-1}^{-\frac{1}{2}}(f_t),
\]
and hence
\[
F_2(f_t) \lesssim A^{-\frac{(d-6)}{3d}} M_d^2(f_t) I_{-1}(f_t) + A^{\frac{d-6}{6d}} M_d^{-\frac{1}{2}}(f_t) \left( e^A + A^{-\frac{d-6}{2d}} M_d^3(f_t) \right).
\]
Let \(\eta\) be a small number, one can choose \(A\) large enough such that
\[
A^{-\frac{(d-6)}{3d}} M_d^2(f_t) = \eta, \quad \text{i.e.} \quad A = \eta^{\frac{3d}{(d-6)}} M_d^{\frac{6}{(d-6)}}(f_t).
\]
Combining (9),(11), (18), (19) and (20), we have
\[
\frac{d}{dt} H(f_t) + (1 - \eta) I_{-1}(f_t) \lesssim \eta^{\frac{1}{2}} \left( \exp \{ \eta^{\frac{3d}{(d-6)}} M_d^{\frac{6}{(d-6)}}(f_t) \} + \eta^3 \right) + M_d^3(f_t)
\lesssim \exp \{ \eta^{\frac{3d}{(d-6)}} M_d^{\frac{6}{(d-6)}}(f_t) \}.
\]
This implies (using proposition 8)
\[ \int_0^T I_1(f_t) dt \lesssim C_T, \]
where \( C_T \lesssim \exp\{CT^{\kappa}\} \), \( C > 0 \) is a positive constant and \( \kappa = \frac{2(d-2)}{(d-6)} \). This completes the proof of the proposition. \( \square \)

3. Proof of Theorem 1

We now divide the proof of theorem 1 into several steps.

Step 1: Modified entropy production estimate. We apply (8) with \( \beta(x) = (x+1) \log(x+1) \), one easily checks that \( \beta''(x) = \frac{1}{x+1} \) and \( 0 \leq \phi_\beta(x) = x - \log(x+1) \leq x \). Since \( H(f_{in}) < \infty \) by assumption, we easily see that \( \int \beta(f_t(v)) dv < \infty \). The ellipticity of \( \pi^f_t \) in proposition 9 immediately implies
\[ \frac{d}{dt} \int \beta(f_t(v)) dv \]
\[ \leq -C_{coer} \int \langle v \rangle^\gamma \frac{\|\nabla f_t(v)\|^2}{1 + f_t(v)} dv + 2(\gamma + 3) \int \int |v - v_s|^\gamma f_t(v)f_t(v_s) dv dv_s \]
\[ \equiv W_1(f_t) + W_2(f_t). \]

Step 2: Estimate of \( W_1(f_t) \). First,
\[ \int \langle v \rangle^\gamma \frac{\|\nabla f_t(v)\|^2}{1 + f_t(v)} dv = 4 \int \langle v \rangle^\gamma \|\nabla(\sqrt{1 + f_t(v)} - 1)\|^2 dv \]
\[ \geq 4(1 + D)^\gamma \|\nabla(\sqrt{1 + f_t} - 1)\|^2_{L^2(B_D)}, \]
for any \( D > 0 \), where \( B_D = \{ x \in \mathbb{R}^3 : |x| < D \} \). By the Sobolev embedding theorem, there exists a constant \( C_{sob} > 0 \) such that
\[ \|\nabla(\sqrt{1 + f_t} - 1)\|^2_{L^2(B_D)} \geq C_{sob}\|\sqrt{1 + f_t} - 1\|^2_{L^6(B_D)} - \|\sqrt{1 + f_t} - 1\|^2_{L^8(B_D)} \]
\[ \geq C_{sob}\|f_t1_{\{|f_t|\geq1\}}\|^2_{L^6(B_D)} - \|f_t\|_{L^3(B_D)} \]
\[ \geq C_{sob}\|f_t1_{\{|f_t|\geq1\}}\|_{L^3(B_D)} - M_0(f_t) \]
\[ \gtrsim \|f_t1_{\{|f_t|\geq1\}}\|_{L^3(B_D)} - 1. \]

Finally, for all \( D \geq 1 \), since \( (1 + D) \leq 2D \), we have
\[ (22) \quad W_1(f_t) \lesssim -D^\gamma (\|f_t1_{\{|f_t|\geq1\}}\|_{L^3(B_D)} - 1). \]

Remark Basically step 1 and step 2 follow the idea from Desvillettes-Villani [7]. However, if we want to improve Fournier-Guerin’s result in [10], we need some new ideas in the following steps.

Step 3: Estimate of \( W_2(f_t) \). We shall present the estimates of \( W_2(f_t) \) for \( \gamma \in (-2, 0) \) and \( \gamma = -2 \) separately.
where $\frac{-\gamma}{3} + \frac{1}{q_1} + \frac{1}{q_2} = 2$ or $\frac{1}{q_1} + \frac{1}{q_2} = \frac{6 + \gamma}{3}$, if we choose $q_1 = q_2$, then

$$W_2(f_i) \leq C_{\mathrm{har}}\|f_i\|_{L^{q_1}}\|f_i\|_{L^{q_2}},$$

Let us interpolate $L^\frac{\varepsilon}{3+\varepsilon}$ between $L^1$ and $L^{3-\varepsilon}$, i.e. $\frac{6+\gamma}{6} = \frac{x}{3-\varepsilon} + \frac{1-x}{1}$, we have $x = \frac{-\gamma(3-\varepsilon)}{6(2-\varepsilon)}$, and hence

$$W_2(f_i) \lesssim \|f_i\|_{L^{3-\varepsilon}} M_0^\frac{3}{3(2-\varepsilon)} (f_i) \lesssim \|f_i\|_{L^{3-\varepsilon}}.$$

**Case II**: $\gamma = -2$. The estimate of $W_2(f_i)$ for $\gamma = -2$ is similar (may be easier) to the estimate of $F_2(f_i)$ in proposition 11. Actually, we have

$$W_2(f_i) \lesssim I_{-1}(f_i) M_0^\frac{3}{2} (f_i) + M_0^\frac{3}{2} (f_i).$$

Combining (21)–(24), we have

$$\frac{d}{dt} \int \beta(f_i(v))dv \leq -D^\gamma (\|f_i1_{\{f_i \geq 1\}}\|_{L^3(B_D)} - 1) + W_2(f_i),$$

or

$$D^\gamma \int_0^T \|f_i1_{\{f_i \geq 1\}}\|_{L^3(B_D)} dt \leq C(H(f_i)) + \int_0^T W_2(f_i) dt \lesssim 1 + \int_0^T W_2(f_i) dt,$$

where

$$\int_0^T W_2(f_i) dt \lesssim \begin{cases} \int_0^T \|f_i\|_{L^{3-\varepsilon}}^\frac{-\gamma(3-\varepsilon)}{3(2-\varepsilon)} dt, & \gamma \in (-2, 0), \\ (1 + T)^\frac{\gamma}{2} \int_0^T I_{-1}(f_i) dt + (1 + T)^\frac{5}{2}, & \gamma = -2. \end{cases}$$

**Step 4**: Weighted estimate. For $s > 0$, we define the weighted function

$$g_t(v) = \langle v \rangle^{-s} f_t(v)1_{\{f_t(v) \geq 1\}}.$$

Now, we will show some relation between $\|g_t\|_{L^3}$ and $\|f_t1_{\{f_t \geq 1\}}\|_{L^{3-\varepsilon}}$. Using (25),

$$\int_0^T \|g_t\|_{L^3} dt \leq \int_0^T \sum_{k \geq 0} \|g_t\|_{L^3(\{2^k - 1 \leq |v| \leq 2^{k+1} - 1\})} dt$$

$$\leq 2^{-\gamma} \int_0^T \sum_{k \geq 0} 2^{(k+1)\gamma} \|f_t1_{\{f_t \geq 1\}}\|_{L^3(B_{2^{k+1}})} dt$$

$$\lesssim 2^{-\gamma} \sum_{k \geq 0} 2^{-s(k+1)} \left[ 1 + \int_0^T W_2(f_i) dt \right]$$
\[
\lesssim 1 + \int_0^T W_2(f_t) \, dt.
\]
On the other hand, let us decompose the function \(f_t^{3-\varepsilon}(v)\) as
\[
f_t^{3-\varepsilon}(v) = \left( f_t(\langle v \rangle)^{\gamma - s} \right)^{\frac{3}{q_1}} \left( f_t(\langle v \rangle)^q \right)^{\frac{1}{q_2}},
\]
where \(\frac{1}{q_1} + \frac{1}{q_2} = 1\), \(\frac{3}{q_1} + \frac{1}{q_2} = 3 - \varepsilon\) and \(\frac{3(\gamma - s)}{q_1} + q_2 = 0\), then we have \(\frac{1}{q_1} = \frac{2 - \varepsilon}{2}, \frac{1}{q_2} = \frac{\varepsilon}{2}\) and \(q = \frac{-3(\gamma - s)(2 - \varepsilon)}{\varepsilon}\), this implies
\[
\tag{28}
\| f_t \mathbf{1}_{\{f_t \geq 1\}} \|_{L^{3-\varepsilon}} \lesssim M_q^{\frac{\varepsilon}{3(2-\varepsilon)}}(f_t) \| g_t \|_{L^3}^\frac{1}{\gamma},
\]
where \(\alpha = \frac{2(3 - \varepsilon)}{3(2 - \varepsilon)}\).

**Step 5: Conclusion.** Consequently, by (27) and (28),
\[
\int_0^T \| f_t \|_{L^{3-\varepsilon}}^\alpha \, dt \lesssim \int_0^T \| f_t \mathbf{1}_{\{f_t < 1\}} \|_{L^{3-\varepsilon}}^\alpha \, dt + \int_0^T \| f_t \mathbf{1}_{\{f_t \geq 1\}} \|_{L^{3-\varepsilon}}^\alpha \, dt
\]
\[
\leq TM_0^{\frac{\alpha}{3(2-\varepsilon)}}(f_T) + \int_0^T M_q^{\frac{\alpha}{3(2-\varepsilon)}}(f_t) \| g_t \|_{L^3} \, dt
\]
\[
\lesssim T + M_q^{\frac{\alpha}{3(2-\varepsilon)}}(f_T) \left[ 1 + \int_0^T W_2(f_t) \, dt \right].
\]
We shall present the conclusion for \(\gamma \in (-2, 0)\) and \(\gamma = -2\) separately.

**Case I:** \(\gamma \in (-2, 0)\). Let \(\delta\) be a parameter to be chosen later, by (26), (29) and the Young inequality,
\[
\int_0^T \| f_t \|_{L^{3-\varepsilon}}^\alpha \, dt \lesssim T + M_q^{\frac{\alpha}{3(2-\varepsilon)}}(f_T) \left[ 1 + \int_0^T \| f_t \|_{L^{3-\varepsilon}}^{-\gamma(3-\varepsilon) \frac{\alpha}{3(2-\varepsilon)}} \, dt \right]
\]
\[
\lesssim M_q^{\frac{\alpha}{3(2-\varepsilon)}}(f_T) \left[ 1 + \int_0^T \left\{ \left( \delta \| f_t \|_{L^{3-\varepsilon}}^{-\gamma(3-\varepsilon) \frac{\alpha}{3(2-\varepsilon)}} \right)^{\frac{\gamma}{\gamma(3-\varepsilon)}} + \delta \frac{2}{\gamma(3-\varepsilon)} \right\} \, dt \right].
\]
Let \(\eta\) be a small number, one can choose \(\delta\) such that
\[
M_q^{\frac{\alpha}{3(2-\varepsilon)}}(f_T) \delta^{-\frac{2}{\gamma(3-\varepsilon)}} = \eta, \quad \text{i.e.} \quad \delta = \eta^{-\frac{\gamma}{2}} M_q^{\frac{\gamma \alpha}{3(2-\varepsilon)}}(f_T).
\]
This means
\[
\int_0^T \| f_t \|_{L^{3-\varepsilon}}^\alpha \, dt \lesssim C_T + \eta \int_0^T \| f_t \|_{L^{3-\varepsilon}}^\alpha \, dt,
\]
where
\[
C_T \lesssim M_q^{\frac{\alpha}{3(2-\varepsilon)}}(f_T) + T \eta^{\frac{\gamma}{(2+\gamma)(3-\varepsilon)}} M_q^{\frac{\gamma \alpha}{(2+\gamma)(3-\varepsilon)}}(f_T) \lesssim TM_q^{\frac{\gamma \alpha}{(2+\gamma)(3-\varepsilon)}}(f_T) \lesssim (1 + T)^{1+ \frac{2}{\gamma(3-\varepsilon)(2+\gamma)}}.
\]
Case II: \( \gamma = -2 \). By (26) and (29),
\[
\int_0^T \| f_t \|^\alpha_{L^{3-\varepsilon}} dt \lesssim T + M_q^{\frac{\alpha\varepsilon}{3-\varepsilon}} (f_T) \left[ 1 + \int_0^T W_2(f_t) dt \right] 
\lesssim M_q^{\frac{\alpha\varepsilon}{3-\varepsilon}} (f_T) \left[ (1 + T)^{\frac{3}{2}} \int_0^T I_{-1}(f_t) dt + (1 + T)^{\frac{5}{3}} \right].
\]

Note that \( q = \frac{3(2+s)(2-\varepsilon)}{\varepsilon} > 6 \), by proposition 11, we have
\[
\int_0^T \| f_t \|^\alpha_{L^{3-\varepsilon}} dt \lesssim \exp \left\{ C T^z \right\},
\]
for some constant \( C > 0 \) and
\[
z = \frac{2(q - 2)}{(q - 6)} = 2 + \frac{8\varepsilon}{3(2 + s)(2 - \varepsilon) - 6\varepsilon}.
\]
This completes the proof of theorem 1. \( \square \)

4. PROOF OF THEOREM 2

We divide the proof of theorem 2 into several steps.

Step 1: Energy estimate. We apply (8) with the function \( \beta(x) = \frac{1}{p} x^p \),
\[
(30) \quad \frac{d}{dt} \frac{1}{p} \| f_t \|_{L^p}^p + \mathcal{B}(f_t) = 0,
\]
where
\[
\mathcal{B}(f_t) = (p - 1) \int \int a(v - v_*) f_t(v_*) f_t^{(p-2)}(v) \nabla f_t(v) \nabla f_t(v) dv dv_*
\]
\[
- \frac{1}{p} \int \int a(v - v_*) \nabla_* f_t(v_*) \nabla f_t^p(v) dv dv_*
\]
\[
\equiv B_1(f_t) - B_2(f_t).
\]
More precisely,
\[
B_1(f_t) = (p - 1) \int \pi^{\varepsilon}(v) f_t^{(p-2)}(v) |\nabla f_t(v)|^2 dv,
\]
and
\[
(33) \quad B_2(f_t) = \frac{1}{p} \int \bar{\pi}^{\varepsilon}(v) f_t^p(v) dv.
\]

Step 2: Estimate of \( B_1(f_t) \). The ellipticity of \( \pi^{\varepsilon} \) in proposition 9 immediately implies
\[
B_1(f_t) \gtrsim \int \langle v \rangle^{\gamma} f_t^{(p-2)}(v) |\nabla f_t(v)|^2 dv
\]
\[
= \frac{4}{p^2} \langle v \rangle^{\frac{3}{2}} \nabla f_t^2 \|_{L^2}^2
\]
\[
\gtrsim \| \nabla (\langle v \rangle^{\frac{3}{2}} f_t^2 \|_{L^2}^2 - \| f_t \|_{L^p}^p.
\]
Step 3: Estimate of $B_2(f_i)$. We further decompose $B_2(f_i)$ as

$$B_2(f_i) = \frac{1}{p} \int \bar{c}^t(v)f_t^p(v)dv$$

$$\lesssim \int \int |v - v_*|^\gamma f_t^p(v)f_t(v_*)dvdv_*$$

$$= \int \int_{|v - v_*| > 1} + \int \int_{|v - v_*| \leq 1} |v - v_*|^\gamma f_t^p(v)f_t(v_*)dvdv_*$$

$$\equiv B_{21}(f_i) + B_{22}(f_i).$$

It is easy to see that

$$B_{21}(f_i) \leq M_0(f_i)\|f_i\|_{L^p}^p \lesssim \|f_i\|_{L^p}^p. \tag{36}$$

We shall present the estimates of $B_{22}(f_i)$ for $\gamma \in (-2, 0)$ and $\gamma = -2$ separately.

Case I: $\gamma \in (-2, 0)$. Note that $\langle v_*\rangle^{\gamma}\langle v\rangle^{-\gamma} \leq C$ if $|v - v_*| \leq 1$, we have obtained

$$B_{22}(f_i) \leq \int_{v_*} \int_{|v - v_*| \leq 1} |v - v_*|^\gamma \langle v_*\rangle^{-\gamma} f_t(v_*) \left(\langle v\rangle^{\frac{\gamma}{2}} f_t^\frac{p}{2}(v)\right)^2 \langle v_*\rangle^{\gamma} \langle v\rangle^{-\gamma} dv_2dv_*$$

$$\lesssim \int_{v_*} \langle v_*\rangle^{-\gamma} f_t(v_*) \int_v |v - v_*|^\gamma \left(\langle v\rangle^{\frac{\gamma}{2}} f_t^\frac{p}{2}(v)\right)^2 dv_2dv_*.$$

We apply the Pitt inequality [3, 4, 5] to get that

$$\int_v |v - v_*|^\gamma \left(\langle v\rangle^{\frac{\gamma}{2}} f_t^\frac{p}{2}(v)\right)^2 dv \leq c_{\text{pitt}} \int_\xi |\xi|^{-\gamma} \left(\langle v\rangle^{\frac{\gamma}{2}} f_t^\frac{p}{2}(v)\right)^2 d\xi,$$

then (37) becomes

$$B_{22}(f_i) \lesssim M_{-\gamma}(f_t) \int_\xi |\xi|^{-\gamma} \left(\langle v\rangle^{\frac{\gamma}{2}} f_t^\frac{p}{2}(v)\right)^2 d\xi$$

$$\lesssim \int_{|\xi| < R_*} + \int_{|\xi| \geq R_*} |\xi|^{-\gamma} \left(\langle v\rangle^{\frac{\gamma}{2}} f_t^\frac{p}{2}(v)\right)^2 d\xi$$

$$\lesssim \widetilde{B}_{221}(f_i) + \widetilde{B}_{222}(f_i),$$

where $R_*$ is a large number to be chosen later. We shall present the estimates of $\widetilde{B}_{221}(f_i)$ for $1 < p < 2$ and $p \geq 2$ separately. If $1 < p < 2$, by the Parseval theorem,

$$\widetilde{B}_{221}(f_i) \lesssim R_*^{-\gamma} \|\langle v\rangle^{\frac{\gamma}{2}} f_t^\frac{p}{2}(v)\|_{L^2}^2 \lesssim R_*^{-\gamma} \|f_t\|_{L^p}^p \lesssim R_*^{-\gamma + 3} \|f_t\|_{L^p}^p. \tag{39}$$

If $p \geq 2$, let us interpolate $L^\frac{p}{2}$ between $L^1$ and $L^p$, we obtain

$$\widetilde{B}_{221}(f_i) \lesssim R_*^{-\gamma + 3} \|\langle v\rangle^{\frac{\gamma}{2}} f_t^\frac{p}{2}(v)\|_{L^\frac{p}{2}}^2 \lesssim R_*^{-\gamma + 3} \|f_t\|_{L^1}^2 \lesssim R_*^{-\gamma + 3} \|f_t\|_{L^p}^\frac{p}{2} \lesssim R_*^{-\gamma + 3} \|f_t\|_{L^p}^{(p-2)/(p-1)}. \tag{40}$$

This means for all $1 < p < \infty$,

$$\widetilde{B}_{221}(f_i) \lesssim R_*^{-\gamma + 3} (1 + \|f_t\|_{L^p}^p). \tag{41}$$
For $\tilde{B}_{222}(f_t)$, we have
\begin{equation}
\tilde{B}_{222}(f_t) \lesssim R_*^{-\gamma - 2} \int_{|\xi| \geq R_*} |\xi|^2 \langle v \rangle^{\gamma} \hat{f}_t^\xi(v) \rangle^2 d\xi \lesssim R_*^{-\gamma - 2} \| \nabla (\langle v \rangle^{\frac{\gamma}{2}} f_t^\xi) \|_{L^2}^2.
\end{equation}
Combining (41)–(42), we get
\begin{equation}
B_{22}(f_t) \lesssim R_*^{-\gamma + 3} (1 + \| f_t \|_{L^p}^p) + R_*^{-\gamma - 2} \| \nabla (\langle v \rangle^{\frac{\gamma}{2}} f_t^\xi) \|_{L^2}^2.
\end{equation}
Using (36) and (43), we have the estimate for $B_2(f)$,
\begin{equation}
B_2(f_t) \lesssim \| f_t \|_{L^p}^p + R_*^{-\gamma + 3} (1 + \| f_t \|_{L^p}^p) + R_*^{-\gamma - 2} \| \nabla (\langle v \rangle^{\frac{\gamma}{2}} f_t^\xi) \|_{L^2}^2.
\end{equation}

Case II: $\gamma = -2$. We further decompose $B_{22}(f_t)$ as
\begin{equation}
B_{22}(f_t) = \int \int_{|v - v_*| \leq 1} |v - v_*|^{-2} f_t^\xi(v) f_t(v_*) \{ 1_{\{ \log f_t(v_*) \geq A \}} + 1_{\{ \log f_t(v_*) < A \}} \} dv dv_*
\equiv B_{221}(f_t) + B_{222}(f_t),
\end{equation}
where $A = A(t)$ is a large parameter to be chosen later. It is easy to see that
\begin{equation}
B_{222}(f_t) \leq e^A \| f_t \|_{L^p}^p.
\end{equation}
For the difficult part $B_{221}(f_t)$, note that $|v - v_*| \leq 1$, we have $\langle v_* \rangle^{-2} \langle v \rangle^2 \leq 3$, and hence
\begin{equation}
B_{221}(f_t) \leq \int v_* \int_{|v - v_*| \leq 1} |v - v_*|^{-2} \langle v_* \rangle^2 f_t(v_*) 1_{\{ \log f_t(v_*) \geq A \}} \langle \langle v \rangle^{-1} f_t^\xi(v) \rangle \langle \langle v \rangle^{-2} f_t^\xi(v) \rangle^2 dv dv_*
\lesssim \int \langle v_* \rangle^2 f_t(v_*) 1_{\{ \log f_t(v_*) \geq A \}} \int v - v_* \langle \langle v \rangle^{-1} f_t^\xi(v) \rangle^2 dv dv_*.
\end{equation}
We apply the Pitt inequality to get that
\begin{equation}
\int v - v_* \langle \langle v \rangle^{-1} f_t^\xi(v) \rangle^2 dv \leq c_{\text{pitt}} \int |\xi|^2 \langle \langle v \rangle^{-1} f_t^\xi(v) \rangle^2 d\xi,
\end{equation}
then
\begin{equation}
B_{221}(f_t) \lesssim \int v_* \int \langle v_* \rangle^2 \langle v \rangle f_t(v_*) 1_{\{ \log f_t(v_*) \geq A \}} dv_* \int |\xi|^2 \langle \langle v \rangle^{-1} f_t^\xi(v) \rangle^2 d\xi
\equiv B_{2211}(f_t) B_{2212}(f_t).
\end{equation}
We now analyze the term $B_{2211}(f_t)$, note that $\log f_t(v_*) \geq A$, we can decompose $\langle v_* \rangle^2 f_t(v_*)$ as
\begin{equation}
\langle v_* \rangle^2 f_t(v_*) \lesssim (\langle v_* \rangle^{2q_1} f_t(v_*))^{\frac{1}{q_1}} (f_t(v_*) \log f_t(v_*))^{\frac{1}{q_2}} A_{\frac{1}{q_2}},
\end{equation}
where $\frac{1}{q_1} + \frac{1}{q_2} = 1$, one can choose $2q_1 = \lambda, \frac{1}{q_2} = 1 - \frac{2}{\lambda} \lambda > 2$, hence the Hölder inequality implies
\begin{equation}
B_{2211}(f_t) \leq A^{-\frac{(\lambda - 2)}{\lambda}} H^{\frac{(\lambda - 2)}{\lambda}} (f_t) M_{\lambda}^2 (f_t) \lesssim A^{-\frac{(\lambda - 2)}{\lambda}} M_{\lambda}^2 (f_t).
\end{equation}
Next for $B_{2212}(f_t)$, 

$$B_{2212}(f_t) = \| \nabla (\langle v \rangle^{-1} f_t^2) \|_{L^2}^2.$$ 

Combining (36), (46) and (47)-(49), we have 

$$B_2(f_t) \lesssim A^{-\frac{(\lambda-2)}{\lambda}} M_\lambda^2(f_t) \| \nabla (\langle v \rangle^{-1} f_t^2) \|_{L^2}^2 + \| f_t \|_{L^p}^p + e^A \| f_t \|_{L^p}^p.$$ 

**Step 4: Conclusion.** We shall present the conclusion for $\gamma \in (-2, 0)$ and $\gamma = -2$ separately.

**Case I:** $\gamma \in (-2, 0)$. Consequently, by (30), (31), (34) and (44) 

$$\frac{d}{dt} \| f_t \|_{L^p}^p + (1 - R_*^{-\gamma - 2}) \| \nabla (\langle v \rangle^{-1} f_t^2) \|_{L^2}^2 \lesssim R_*^{-\gamma + 3} (1 + \| f_t \|_{L^p}^p).$$ 

One can choose $R_*$ large enough, then it is easy to get 

$$\| f_t \|_{L^p}^p \lesssim e^{Ct},$$ 

for some constant $C > 0$.

**Case II:** $\gamma = -2$. By (30), (31), (34) and (50) 

$$\frac{d}{dt} \| f_t \|_{L^p}^p + \| \nabla (\langle v \rangle^{-1} f_t^2) \|_{L^2}^2 \lesssim A^{-\frac{(\lambda-2)}{\lambda}} M_\lambda^2(f_t) \| \nabla (\langle v \rangle^{-1} f_t^2) \|_{L^2}^2 + \| f_t \|_{L^p}^p + e^A \| f_t \|_{L^p}^p.$$ 

Let $\eta_1 > 0$ be a small number, one can choose $A = A(t)$ large enough such that 

$$A^{-\frac{(\lambda-2)}{\lambda}} M_\lambda^2(f_t) = \eta_1,$$ 

or say (using proposition 8) 

$$A(t) = \eta_1^{\frac{-\lambda}{\lambda - 2}} M_\lambda^{\frac{2}{\lambda - 2}}(f_t) \lesssim \begin{cases} \eta_1^{\frac{\lambda}{\lambda - 2}} (1 + t)^{\frac{2}{\lambda - 2}} & \text{if } 2 < \lambda < 5, \\ \eta_1^{\frac{-\lambda}{\lambda}} (1 + t)^{\frac{2}{\lambda}} & \text{if } \lambda \geq 5. \end{cases}$$ 

then (52) becomes 

$$\frac{d}{dt} \| f_t \|_{L^p}^p + (1 - \eta_1) \| \nabla (\langle v \rangle^{-1} f_t^2) \|_{L^2}^2 \lesssim \| f_t \|_{L^p}^p + e^A \| f_t \|_{L^p}^p.$$ 

If $f_m(v) \in L^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, $2 < \lambda < 5$, then (54) implies 

$$\| f_t \|_{L^p}^p \lesssim t \exp \{ A(t) \} \lesssim \exp \{ Ct^{\frac{2}{\lambda - 2}} \}.$$ 

On the other hand, if $\lambda \geq 5$, we have 

$$\| f_t \|_{L^p}^p \lesssim t \exp \{ A(t) \} \lesssim \exp \{ Ct^{\frac{2}{\lambda}} \}.$$ 

However, if $f_m(v) \in L^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, $\lambda > 6$, then we can improve the temporal growth. Let us prove the case $1 < p \leq 2$ first. Consider the following decomposition: 

$$f_t^p(v) = (\langle v \rangle^{-1} f_t^p(v))^{\frac{p}{n_1}} (\langle v \rangle^{-1} f_t^\frac{p}{n_2})^{\frac{p}{n_2}} (\langle v \rangle^a f_t(v))^{\frac{1}{n_1}},$$
where \(\frac{3p}{q_1} + \frac{3}{q_2} + \frac{1}{q_3} = p, \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1\) and \(\frac{6}{q_1} + \frac{6}{q_2} + \frac{a}{q_3} = 0\), one can choose \(\frac{1}{q_1} = \frac{1}{q_2} = \frac{p}{3p+1}\), \(\frac{1}{q_3} = \frac{p+3}{3p+1}\) and \(a = \frac{12(p-1)}{p+3}\), by the Hölder inequality and the Sobolev embedding theorem,

\[
\|f_t\|_{L^p}^p \leq \|\langle v \rangle^{-1} f_t \|_{L^6}^{\frac{6(p-1)}{3p+1}} \|\langle v \rangle^{-1} f_t \|^2_{L^2} M_{a}^{\frac{p+3}{3p+1}} (f_t)
\]

\[
\lesssim \left( \| \nabla (\langle v \rangle^{-1} f_t^p) \|_{L^2}^2 + \| f_t \|_{L^p}^{3(p-1)} (I_1 - 1) + 1 \right) M_{a}^{\frac{p+3}{3p+1}} (f_t),
\]

and hence by the Young inequality, there exists a small number \(\eta_2 > 0\) such that

\[
e^A \| f_t \|_{L^p}^p \lesssim e^{\frac{3p+1}{3p+7} A M_{a}^{\frac{p+3}{3p+1}}} (f_t) + \eta_2^{-1} (I_1 - 1) + 1 + \eta_2 \left( \| \nabla (\langle v \rangle^{-1} f_t^p) \|_{L^2}^2 + \| f_t \|_{L^p}^p \right).
\]

Combining (52), (53) and (55), we have

\[
\frac{d}{dt} \| f_t \|_{L^p}^p + (1 - \eta_1 - \eta_2) \| \nabla (\langle v \rangle^{-1} f_t^p) \|_{L^2}^2
\]

\[
\lesssim \| f_t \|_{L^p}^p + M_{a}^{\frac{p+3}{3p+7}} (f_t) \exp \left\{ \frac{3p+1}{-3p+7} \frac{1}{\eta_1} \right\} + \eta_2^{-1} I_1 (f_t) + 1,
\]

using the estimate of the weighted Fisher information in proposition 11, we have

\[
\| f_t \|_{L^p}^p \lesssim C_t,
\]

where

\[
C_t \lesssim \exp \left\{ C t^{\frac{2(\lambda-2)}{(\lambda-6)}} \right\},
\]

\(C > 0\) is a positive constant. This completes the case \(1 < p \leq 2\). For \(p > 2\), we will prove the following argument: let \(p \geq 2, 0 < \delta \leq 1\) and the initial condition \(f_{in} \in L^{p+\delta}\), if \(\| f_t \|_{L^p}^p \lesssim \exp \left\{ C t^{\frac{2(\lambda-2)}{(\lambda-6)}} \right\}\), then \(\| f_t \|_{L^{p+\delta}} \lesssim \exp \left\{ C t^{\frac{2(\lambda-2)}{(\lambda-6)}} \right\}\), where \(C \leq C'\). If one can prove this argument, then the bootstrap procedure finishes the proof of the theorem.

**Proof of the argument.** For the \(L^{p+\delta}\) norm, note that (52) can be rewritten as

\[
\frac{d}{dt} \| f_t \|_{L^{p+\delta}} + (1 - \eta_1) \| \nabla (\langle v \rangle^{-1} f_t^{(p+\delta)}) \|_{L^2}^2 \lesssim \| f_t \|_{L^{p+\delta}} + e^A \| f_t \|_{L^{p+\delta}}.
\]

Again, we need more analysis for \(e^A \| f_t \|_{L^{p+\delta}}^p\). Consider the following decomposition:

\[
f_t^{p+\delta} (v) = (\langle v \rangle^{-1} f_t \frac{(p+\delta)}{2} (v)) \frac{6}{q_1} (f_t^p (v)) \frac{1}{2} (\langle v \rangle^d f_t (v)) \frac{1}{q_1},
\]

where \(\frac{3(p+\delta)}{q_1} + \frac{2}{q_2} + \frac{1}{q_3} = p + \delta, \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1\) and \(\frac{-6}{q_1} + \frac{d}{q_3} = 0\), one can choose \(\frac{1}{q_1} = \frac{1}{6}\), \(\frac{1}{q_2} = \frac{3(p+\delta)-5}{6(p-1)}, \frac{1}{q_3} = \frac{2p-3d}{6(p-1)}\) and \(d = q_3\), by the Hölder inequality, there exists a small number \(\eta_2 > 0\) such that

\[
e^A \| f_t \|_{L^{p+\delta}} \lesssim e^A \| \langle v \rangle^{-1} f_t \|_{L^6}^2 \| f_t \|_{L^2}^{\frac{p}{q_2}} M_{a}^{\frac{1}{q_2}} (f_t)
\]

\[
\lesssim \eta_2 \| \langle v \rangle^{-1} f_t \|_{L^6}^2 + \eta_2^{-1} e^{2A} \| f_t \|_{L^2}^{\frac{2p}{q_2}} M_{a}^{\frac{1}{q_2}} (f_t),
\]

(57)
\[ \lesssim \eta_2 \| \nabla (\langle v \rangle^{-1}_t f_t^{(p+\delta)}) \|_{L^2}^2 + \eta_2 \| f_t \|_{L^{p+\delta}}^{p+\delta} + \eta_2^{-1} (1 + t)^{\frac{\lambda}{2}} \exp \left\{ 2A + Ct^2 \frac{(\lambda-2)}{(\lambda-6)} \right\}, \]

Combining (56)–(57), we have

\[ \frac{d}{dt} \| f_t \|_{L^{p+\delta}}^{p+\delta} + (1 - \eta_1 - \eta_2) \| \nabla (\langle v \rangle^{-1}_t f_t^{(p+\delta)}) \|_{L^2}^2 \lesssim \| f_t \|_{L^{p+\delta}}^{p+\delta} + \exp \left\{ 2\eta_1^{\frac{\lambda}{2}} t^{\frac{\lambda}{2}} + Ct^{\frac{(\lambda-2)}{(\lambda-6)}} \right\}, \]

hence

\[ \| f_t \|_{L^{p+\delta}}^{p+\delta} \lesssim C_t, \]

where

\[ C_t \lesssim \exp \left\{ C't^2 \frac{(\lambda-2)}{(\lambda-6)} \right\}. \]

This completes the proof of theorem 2. \( \square \)

References

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